Partial Differential Equations

* Algebraic Eqn

An equation obeyed by a number \( x \), e.g.

\[ 2x - 6 = 0 \quad \Rightarrow \quad x = 3 \]

\[ ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Goal: find number(s) obeying the eqn

~ Diophantine: restricted to integers, e.g. pythagorean triples.

* Differential Eqn:

An equation obeyed by a function \( x(t) \) of a single variable

\[ m \frac{d^2 x(t)}{dt^2} = -kx \]

Goal: find functions obeying the differential eqn

\[ x(t) = A \sin \sqrt{\frac{k}{m}} t + B \cos \sqrt{\frac{k}{m}} t \]
"Degree" of algebraic eqn $\Rightarrow$ # of solns

\[ ax + b = 0 : \ \text{degree 1} \Rightarrow 1 \ \text{solution} \]

\[ ax^2 + bx + c = 0 : \ \text{degree 2} \Rightarrow 2 \ \text{solutions} \]

etc.

Same for differential equation: Count # of derivatives

First derivative
\[ \frac{dx}{dt} = 4x \quad x(t) = Ae^{4t} \]
- degree 1

A one solution

degree 2
\[ m \frac{d^2x}{dt^2} = -kx \]

Two solutions
Vibrating string \( y(x,t) \)

Wave eqn

\[
\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} = 0
\]

Laplace eqn (later)

Schrodinger Eq. (later)

Diffusion eqn

\[
\frac{\partial p(x,t)}{\partial t} = D \frac{\partial^2 p}{\partial x^2}
\]

\( p(x,t) \) = density of smoke

\( p(x,t) \) = Temperature

Diffusion constant

Large \( D \) \( \rightarrow \) Rapid spreading
Numerical first derivative

\[ \frac{df}{dx} = \frac{f(x+dx) - f(x)}{dx} \]

Second derivative

\[ \frac{d^2f}{dx^2} = \frac{d}{dx} \left( \frac{df}{dx} \right) = \frac{dy}{dx} \left( \frac{df}{dx} \right) \]

\[ = \frac{1}{dx} \left[ \frac{f(x+dx) - f(x)}{dx} - \frac{f(x) - f(x-dx)}{dx} \right] \]

\[ = \frac{f(x+dx) - 2f(x) + f(x-dx)}{dx^2} \]

Check

\[ f(x) = mx + b \]

\[ f(x+dx) = mx + mdx + b \]

\[ f(x-dx) = mx - mdx + b \]

\[ f(x+dx) - 2f(x) + f(x-dx) = 0 \]

Check

\[ f(x) = ax^2 \]

\[ f(x+dx) = a(x^2 + 2xdx + dx^2) \]

\[ f(x-dx) = a(x^2 - 2xdx + dx^2) \]

\[ f(x+dx) - 2f(x) + f(x-dx) = 2a \cdot dx^2 \rightarrow \frac{d^2f}{dx^2} = 2a \]
Alternate derivation (Taylors TM)

\[ f(x+dx) = f(x) + \frac{df}{dx} dx + \frac{1}{2} \frac{d^2 f}{dx^2} (dx)^2 + \cdots \]

\[ f(x-dx) = f(x) - \frac{df}{dx} dx + \frac{1}{2} \frac{d^2 f}{dx^2} (dx)^2 + \cdots \]

\[ f(x+dx) + f(x-dx) = 2f(x) + \frac{d^2 f}{dx^2} dx^2 \quad \Box \]

Same!

Back to

Diffusion Eqn

Partial derivative: change one variable, keep other fixed!

\[ \frac{\partial (x, t+dt) - p(x,t)}{dt} = D \frac{\partial^2 p(x,t)}{dx^2} \]

\[ \frac{\partial}{\partial t} \quad x \text{ fixed, } t \text{ varies} \quad \frac{\partial^2}{\partial x^2} \quad t \text{ fixed, } x \text{ varies} \]

\[ p(x, t+dt) = p(x,t) + \left( \frac{D}{dx^2} \right) (p(x+dx,t) - 2p(x,t) + p(x-dx,t)) \]

\[ + \quad t \]

\[ \frac{\sqrt{2}}{\sqrt{\text{box}}} \]

Picture explains factor of 2 and also signs!