1. Converting \( \frac{\sum_k}{k} \) into \( \int d^3k \), since \( k \) space points are separated by \( \left( \frac{2\pi}{L} \right)^3 \), each point occupies a volume \( \left( \frac{2\pi}{L} \right)^3 \). Thus, a volume \( SL \) in \( k \) space contains \( \frac{1}{(2\pi)^3} \) \( k \) points.

\[
\frac{\sum_k}{k} = \frac{\int (2\pi)^3}{(2\pi)^3} \frac{\int d^3k}{\int d^3k} = \int d^3k
\]

\[
\Sigma^k = \pi^d \text{ Dimensionally correct also}
\]

2. The density of states is defined as the number of states per unit energy.

\[
N(E) = 2 \sum_k \delta(E - E(k)) = 2 \frac{\sqrt{\pi}}{(2\pi)^3} \int \delta(E - E(k))
\]

\[
\text{Spin sum over all \( k \)}
\]

3. Example: In 3-d with \( E(k) = \hbar^2 k^2 / 2m \)

\[
N(E) = \frac{1}{(2\pi)^3} \int_0^\infty \frac{k^2}{2m} \delta \left( E - \frac{\hbar^2 k^2}{2m} \right)
\]

Note: \( \delta(ax) = \frac{1}{a} \delta(x) \)

Proof:

\[
\int f(x) \delta(ax) dx = \int f\left( \frac{y}{a} \right) \delta(y) \frac{dy}{a} = \frac{1}{a} \int f\left( \frac{y}{a} \right) \delta(y) dy
\]

\[
y = ax = 1/2 f(a)
\]
\[ \delta \left( E - \frac{\hbar^2 k^2}{2m} \right) = \delta \left( \frac{\hbar^2}{2m} \left( \frac{2mE}{\hbar^2} - k^2 \right) \right) \]

\[ f(k) = \frac{1}{\Delta m} \]

\[ |f'(k)| = \frac{\hbar^2 k}{2m} \]

\[ \frac{\hbar^2 k_0^2}{2m} = E \]

\[ \delta \left( E - \frac{\hbar^2 k_0^2}{2m} \right) = \frac{1}{\hbar^2 k_0} \delta(k - k_0) \]

\[ N(E) = \frac{V}{\pi^2} \frac{m}{\hbar^2} k_0 \]

\[ = k_0 \sim E^{1/2} \]

\[ N(E) = \frac{V}{\pi^2} \left( \frac{m}{2\pi^2 E} \right)^{1/2} \frac{2mE}{\hbar^2} \sim E^{1/2} \]

\[ \frac{N(E)}{\hbar^2} \]

Physically: Phase space states of energy \( E \) lie a surface of sphere, area \( \sqrt{2nE} \). Why does \( N(E) \) not increase linearly with \( E \)?

Answer is \( dE/dk \), factor \( E \) is changing more rapidly with \( k \) out near here also.
More precisely, suppose $E(k)$.

\[
\delta \left( E - E(k) \right) = \frac{1}{\sqrt{2\pi \hbar^2}} e^{-\frac{\hbar^2}{2m} (k - k_0)^2} \delta \left( k - k_0 \right)
\]

This tells us an important principle: whenever the dispersion relation $E(k)$ is flat, $\frac{dE}{dk} = 0$, $N(E)$ could have a big peak!

Such a peak is called a Van Hove singularity.

In 3-d, $E(k) = \frac{\hbar^2 k^2}{2m}$ and are "softened" by Van Hove by phase space factor $k^3$.

Van Hove singularities are important because whenever $N(E)$ is large $\Rightarrow$ lots of states of hot energy $\Rightarrow$ by response there (Energy equation) consider a scattering experiment...

in fact this is one way $N(E)$ is probed.
Last time we introduced

\[ N(E) = 2 \sum_k \delta(E - E(k)) \]

The density of states. Since in any scattering experiment any energy lost must go into some energy level of the solid, it is important to know the distribution of levels.

We computed \( N(E) \) for \( E(k) = \frac{k^2}{2m} \) in 3d

\[
N(E) = \begin{cases} 
\frac{V}{\pi^2} \frac{2m}{h^2} \left( \frac{2mE}{h^2} \right)^{1/2} & E > 0 \\
0 & E < 0
\end{cases}
\]

\[ N(E_F) = \frac{3N}{E_F} \quad \text{units 1/E as expected} \]

is a handy result obtained by using

\[ E_F = \frac{k_F^2}{2m} \quad N = V \frac{k_F^3}{3\pi^2} \]

\[
\frac{3N}{E_F} = \frac{\pi V k_F^3}{3\pi^2} = V \frac{2m}{\pi^2} \frac{k_F}{h^2} = \frac{\pi^2}{2m} \frac{2m}{h^2} \left( \frac{2mE_F}{h^2} \right)^{1/2}
\]

\# In particular peaks in \( N(E) \) may show up as peaks in scattering rate; gaps in \( N(E) \) will show up as regions of little scattering.
We also noted that in computing $W(E)$

we get $\int \frac{dE}{dh}$ out in limit.

from solving around with $S$-functions - places

$dE/dh$ vanishes or have peaks in $N(E)$

van Hove singularities
Why can we ignore interactions if so many electrons are around? Doesn't high density $\propto$ interaction must dominate? L

No! KE also increases with density and faster.

$$P.E \sim \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r} \sim \frac{1}{4\pi\varepsilon_0} e^2 \frac{1}{n^{1/3}}$$

typical: $n = \frac{\text{number of particles}}{\text{volume}} = \frac{N}{V} = \frac{1}{r^3}$

$$V \sim n r^3$$

But we just saw

$$KE \sim n^{2/3}$$

$P.E.$ grows faster with $n$ than $P.E.$

Wigner crystallization. A collection of $e^{-}$ forms a lattice at low density where $P.E.$ dominates.
Neutron star N neutrons

\[ PE \sim -\frac{9m^2}{r^2} N^2 \sim -\frac{9m^2 N^2}{V^{1/3}} \]

\[ KE \sim +\frac{N}{V} EF. = N \frac{\hbar^2}{2m} \left( \frac{3\pi^2 N}{V^2} \right)^{2/3} \]

\[ \sim \frac{\hbar^2}{2m} \left( \frac{3\pi^2}{V^2} \right)^{2/3} N^{5/3} V^{-2/3} \]

If \( V \rightarrow V/8 \) PE increases by \( x^2 \)

KE increases by \( x^4 \)

Balance between 2 \( \rightarrow \) neutron skin radius

Need to know \( N \), typical \# of neutrons

Balance shown on graph.

seen before

where?

optimal radius
Neutron Star numbers

\[ E = PE + KE \]
\[ = -\frac{\alpha M^2 N^2}{r} + \frac{\hbar^2}{2M} \left(3\pi^2\right)^{2/3} N^{5/3} \frac{1}{r^{12}} \]

\[ r_0 = \frac{\frac{\hbar^2}{6M} \left(3\pi^2\right)^{2/3} N^{5/3}}{G M^3} \]

\[ G = 6.67 \times 10^{-11} \]
\[ M = 1.67 \times 10^{-27} \]
\[ \hbar = 1.055 \times 10^{-34} \]

\[ \Rightarrow N = \frac{2.10^{-36}}{1.67 \times 10^{-27}} \]

\[ \text{M_{Sun}} = 2.0 \times 10^{30} \]

\[ 10^{-68} \]
\[ 16 \]
\[ 10^{11} \]
\[ 10^{81} \]

\[ (1.055)^2 \left(\frac{2}{1.67}\right)^{2/3} \]

\[ \left(6.67 \times 10^{-11}\right)^3 \]

\[ \approx 10^5 \text{ m} \]

\[ \approx 10 \text{ km} \]
Specific heat of solid

\[ c(T) = \gamma T + AT^3 \]

Crude argument

\[ \text{Fermi Energy} \sim 5 \times 10^{-21} J \]

\[ k_B T \sim 4 \times 10^{-21} J \]

\[ \Delta \]

\[ k_x \]

\[ k_y \]

\[ k_z \]

cannot give energy to electron
(deep) inside Fermi sphere because all neighboring states filled (Pauli Blocked)

Only states within \( k_B T \) of \( E_F \) can absorb energy (respond to increase in \( T \))

\[ C \sim N k_B \frac{k_B T}{E_F} \]

\[ \gamma = \frac{C}{T} \]

Classical Pauli
Ideal gas
Blocking reduction

Answer
This is all at $T=0$, assume $e^-$ have minimum possible energy by occupying lowest state possible.

From our review of state much, as $T$ increases, we begin to occupy higher states.

$E_n$ can be occupied or empty, but not multiply occupied.
Example B: A frequently encountered 1-d model is a dispersion relation \( E(k) = -2t \cos k \).

What is \( N(E) \) in such a model?

In 1-d there are no phase space factors, so all we worry about is where \( \frac{dE}{dk} \) vanishes.

\( N(E) \) should have peaks where \( E = \pm 2t \).

Explicit calculation:

\[
N(E) = 2 \frac{L}{2\pi} \int dk \delta(E - 2t \cos k)
\]

Another theorem:

\[
f(k) \quad f'(k) = 2t \sin k
\]

\[
N(E) = \frac{L}{2\pi} \frac{1}{2t \sin k_0} \quad 2t \cos k_0 = E
\]

So \( N(E) \) indeed has a peak

at \( E = \pm 2t \).
Example CP likewise \( \text{e}^{\text{i} \text{hw}} = -2t (\text{c} \text{o} \text{k} \text{e} \text{r + t a} \text{o} \text{k} \text{y}) \)

- 2t (\text{c} \text{o} \text{k} \text{e} \text{r} \text{x} + \text{c} \text{o} \text{k} \text{y} \text{r} \text{o} \text{k} \text{z} \text{e})

in 2-dim. Hubbard model.

HW problem: compute \( N(\text{E}) \) (numerically)?

Kwok pictures

(4?) Usefulness - Again where \( N(\text{E}) \) is by bulk of states to respond to external perturbation.

Analytically, Suppose same occurs even to compute

any quantity that depends on \( N(\text{E}) \) even at finite temperature

\[
\sum_{\text{k}} \left[ \frac{\text{Q}^{(E)}(\text{k})}{\text{e}^{(\text{E}(\text{k}))-\text{M}}+1} \right] = \frac{\text{V}}{(2\pi)^{3}} \int d\text{E}\text{d}\text{k}
\]

\[
\text{V}(\text{E}^{*}) = \int d\text{E} \text{Q}(\text{E}) \frac{1}{\text{e}^{(\text{E}(\text{k}))-\text{M}}+1} \delta(\text{E} - \text{E}(\text{k}))
\]

\[
\frac{1}{\text{k}} = \sum \text{k} \left[ \text{dE} \text{Q}(\text{E}) \frac{1}{\text{e}^{(\text{E}(\text{k}))-\text{M}}+1} \right] \delta(\text{E} - \text{E}(\text{k}))
\]

\[
= \int \text{dE} \text{Q}(\text{E}) \frac{1}{\text{e}^{(\text{E}(\text{k}))-\text{M}}+1} N(\text{E})
\]

(5) A frequent approximation \( N(\text{E}) = N(\text{E}_{\text{F}}) \)

density of states at Fermi surface. Why is that?
\[ \varepsilon_k = -2t(\cos k_x + \cos k_y) \]

### Density of State (2D)

\[ T_C = \frac{\hbar^2}{m} e^{-\frac{1}{\lambda N(E_f)}} \]

\( \lambda N(E_f) \approx \frac{1}{4} \)

\( \omega_D \approx 300 \text{ K} \)

\( 500 e^{-x} = 9.14 \)
\[ \epsilon(k) = -2t (\cos k_x + \cos k_y + \cos k_z) \]

DENSITY OF STATE (3D)
Thermodynamics: Prob of occupying \( e_k \) \( \frac{1}{e^{\beta(E_k - m)} + 1} \)

\[
N = 2 \sum_k \frac{1}{e^{\beta(E_k - m)} + 1} \rightarrow \frac{V}{4\pi^3} \int d^3k \frac{1}{e^{\beta(E_k - m)} + 1}
\]

\[
E = 2 \sum_k E_k \frac{1}{e^{\beta(E_k - m)} + 1} \rightarrow \frac{V}{4\pi^3} \int d^3k E_k \frac{1}{e^{\beta(E_k - m)} + 1}
\]

We evaluated these expressions at \( T = 0 \) when \( \beta = \infty \). Then we took a further step and introduced

\[
N(E) = 2 \sum_k \delta(E - E_k) = \frac{V}{4\pi^3} \int d^3k \delta(E - E_k)
\]

which allowed us to rewrite

\[
N = \left( \frac{V}{4\pi^3} \int d^3k \int dE \delta(E - E_k) \right) \frac{1}{e^{\beta(E - m)} + 1}
\]

\[
= \int dE N(E) \frac{1}{e^{\beta(E - m)} + 1}
\]

where

\[
E = \int dE E N(E) \frac{1}{e^{\beta(E - m)} + 1}
\]

where

\[
N(E) = \frac{N}{k^2} \int \frac{2m}{k^2} - \frac{\sqrt{2mE}}{k^2} \quad \text{for} \quad E = \frac{k^2h^2}{2m}
\]

\[
0 \quad E < 0
\]

Again, we know how to do calculations at \( T = 0 \)

\[
[ ] = \int \frac{dE}{N(E)} \quad \Rightarrow \quad \frac{V}{3 \pi^2} \left( \frac{2m}{h^2} \right)^{\frac{3}{2}}
\]
\[ N(E) = \frac{V}{4\pi^3} \int \frac{d^3k}{\mathcal{V}} \delta \left( E - \frac{\hbar^2 k^2}{2m} \right) \]

\[ f(k) \]
\[ f'(k) = \frac{\hbar^2 k}{2m} \]
\[ E = \frac{\hbar^2 k_0^2}{2m} \]

\[ \frac{2m}{\pi^2} \frac{K_0^2}{h^2-k_0^2} \]

\[ \frac{V}{\pi^2} \frac{2m}{\hbar^2} \left( \frac{2mE}{\hbar^2} \right)^{1/2} \]

Define by taking \( 2V \) from \( n/V \) \( g(E) \)

\[ g(E_F) = \frac{3N}{hE_F} \] is a convenient bound. (A-M)

\[ n(E_F) = 3 \frac{N}{hE_F} \]
Sommerfeld developed a technique for computing these integrals at finite $T$.

$$\int_{-\infty}^{\infty} H(E) \frac{1}{e^{\beta(E - \mu)} + 1} dE$$

$H(E)$ is any function of $E$
$E > E^*(E)$ whenever
$E < E^*(E)$ whenever

$$= \int_{-\infty}^{\infty} H(E) dE + \frac{\pi^2}{6} (k_B T)^2 H'(\mu) + \frac{7\pi^4}{360} (k_B T)^4 H'''(\mu) + \ldots$$

Comments:
(1) Reduces to our $T=0$ result.
(2) Modified

\[ \frac{1}{e^{\beta(E - \mu)} + 1} \]
Applications

\[ N = \int_0^M N(E) dE + \frac{\pi^2}{6} (k_B T)^2 N'(M) + \cdots \]

\[ E = \int_0^M E N(E) dE + \frac{\pi^2}{6} (k_B T)^2 \left[ MN'(M) + N(M) \right] \]

As we will see \( M = E_F + o(k_B T)^2 \) so

\[ \int_0^M N(E) dE = \int_{E_F}^{E_F + N(E)} dE + N(E_F) \left( \frac{1}{2} + \cdots \right) \]

\[ N = \text{old } N \]

Similarly for \( N(E) \). Let\( E \) be it down \( k \)

\[ 0 = N(E_F) \left( \frac{E_F}{2} - E_F \right) + \frac{\pi^2}{6} (k_B T)^2 N'(E_F) \]

\[ \text{so } M = E_F + \frac{\pi^2}{6} (k_B T)^2 \frac{N'(E_F)}{N(E_F)} \]

\[ N(E) = c E^{1/2} \]

\[ N'(E) = \frac{1}{2} c E^{-1/2} \]

\[ \frac{N'(E)}{N(E)} = \frac{1}{2} E \]

\[ M = E_F - \frac{\pi^2}{6} (k_B T)^2 \frac{1}{2} E_F + E_F \left[ 1 - \frac{\pi^2}{12} \left( \frac{k_B T}{E_F} \right)^2 \right] \]

Remember \( E_F = 10^4 - 10^5 \)

\( T \sim 10^2 \)
\[
E = \int_{E_F}^{E_0} \left[ E \frac{dE'}{E'} + (\mu - E_F) E_F N(E_F) \right] + \frac{n^2}{6} (k_B T)^2 \left[ E_F N'(E_F) + N(E_F) \right]
\]

checked terms cancel by eqn for \( N \).

\[
E = E_0 + \frac{n^2}{6} (k_B T)^2 N(E_F)
\]

\[
c = \frac{dE}{dT} = \frac{n^2}{6} (k_B T)^2 k_B N(E_F)
\]

\[
N(E_F) = \frac{3 N}{E_F}
\]

\[
c = \frac{n^2}{6} \frac{(k_B T)^2 k_B}{E_F}
\]

classical \( c = \frac{3}{2} N k_B \) reduced by \( \frac{k_B T}{E_F} \)

\[
\frac{c}{N} = \frac{3}{2} N k_B
\]

Physics of suppression: Pauli prevents

existence of most of electron
* Prediction of linear specific heat is not consequence of FD statistics. Actually as we shall see

\[ C = \gamma T + AT^3 \]

\[ \gamma = \text{phonons} \quad \text{and} \quad \text{electron} \]

Usually plot \( \frac{C}{T} = \gamma + AT^2 \)

We wrote

\[ C = \frac{\pi^2}{30} (k_B T)^3 k_B N(E_F) \quad N(E_F) \sim T^{3/2} \]

Comment on "heavy fermion" materials CeA_2, CeAl_3, ... originally studied why

1) No magnetic order - how come?
2) \( \gamma \) is unusually large - Attributed to
3) Large "m"
4) Sc.

\( \rightarrow \) page 58 \( \rightarrow \)

Filled Fermi Sphere

\[ n=1 \]

\[ n=0 \]

Broadened of surface by \( k_B T \) - these are normal excitations.

What about excitations caused by external probes.
CANONICAL PARTITION FUNCTIONS

classical
(Distinguishable)

\[ N=0 \]
\[ 1 \]

\[ N=1 \]
\[ e^{-\beta E_1} + e^{-\beta E_2} + e^{-\beta E_3} = \sum e^{-\beta E_i} \]

\[ N=2 \]
\[ e^{-2\beta E_1} + e^{-2\beta E_2} + e^{-2\beta E_3} + e^{-\beta E_1} e^{-\beta E_2} + e^{-\beta E_1} e^{-\beta E_3} + e^{-\beta E_2} e^{-\beta E_3} = \sum (e^{-\beta E_i})^2 \]

\[ = (\sum e^{-\beta E_i})^2 \]

\[ (\sum e^{-\beta E_i})^N \]

\[ 2N = 3^N \]

\[ Z = \sum_{N=0}^{\infty} 3^N e^{-\beta N} \]

\[ Z = \prod \left( 1 - e^{-\beta E_i} \right)^{-1} \quad \text{grand canonical} \]

\[ \sum_{N=0}^{\infty} \left[ e^{-\beta E_1} + e^{-\beta E_3} \right]^N \]

\[ = 1 + \sum_{N=1}^{\infty} 3^N e^{-\beta E_1} \cdot e^{-\beta E_2} \cdot e^{-\beta E_3} \]

\[ = 1 + e^{-\beta E_1} + e^{-\beta E_2} + e^{-\beta E_3} \]

\[ + 1 e^{-\beta (E_1 + E_2)} \]

\[ \text{or put in } \mu \text{ b} \]

\[ (1 + e^{-\beta E_1} + e^{-2\beta E_2} + ...) \text{ control filling} \]
I was a bit cavalier in saying scattering \(\Delta N(E)\) only moves 2 conduction bands, create phonons at \(\hbar\).

My mistake is not so precise, must consider energy and momentum.

From the quantum mechanical perspective, this is more precise description of excitation (\(T=0\)).

Knock an electron out of filled Fermi sphere.

**Electron-Hole Pair**

Transfer momentum \(\vec{q}\), change what is energy transfer

\[ E(k+q) = E(k) \]

Thus guy cannot absorb \(\vec{q}\).

Actually, for a given \(\vec{q}\), not any \(E\) is legal. Must go to an unoccupied state.

For any \(\vec{q}\) with \(\vec{q} \neq 2k_F\)

\[ \text{For any } |\vec{q}| < |2k_F| \text{, always find a } \vec{k} \text{ so that } E(k+q) = E(k) \]

We can choose \(\vec{q} \parallel k_F\).

What is the "dispersive relation"? The relation between momentum \(\vec{q}\) transferred and energy harnessed?

A misnomer!

No unique relationship.

For example:

\[ E_{k+q} - E_k = 2k_F q + q^2 \]

\[ q^2 - 2k_F q \]

2k_F \qquad |q|
Suppose $E_{Kq} - E_{k} > 0$. Cannot last unless $|q|$ is sufficiently large. In fact since

$$6E_{Kq} - 6k = 2k \cdot q \cdot q = 2|k||q| \cos \theta + q^2$$

Difference is largest by making $k \parallel q$ and in fact making $|q| = k_f$

$$[E_{Kq} - E_k]_{\text{max}} = 2k_f q + q^2$$

If $|q| > 2k_f$ there is also a minimum energy you can transfer by making $k = -k_f q$

$$[E_{Kq} - E_k]_{\text{min}} = q^2 - 2k_f q$$

Draw picture as we go

"Particle Hole Continuum"

Relevance consider a probe which might transfer energy $\Delta E$ and moving $6E_{Kq}$ past lies outside shaded area cannot lose energy to excitations of el-holes. Maybe some other mechanism within solid

More generally suppose there are other excitations in solid - they will be undamped
We saw one correction & Drude's C = T

method of C as constant also C < (classical)

Purview K also reduced by same factor

so doesn't okay.

Well, effect ok. turns out to be okay.

Perhaps not obvious. QM -- uncertainty principle
as well as Pauli principle when required for duality.

However, E/M fields & Temp gradients justify.

scale of ~10^{-10} A, dp can be quite small.

Except for x-rays.

Drude also assumes AX < A = \sqrt{2} E

Fortunately, \sqrt{2} ~ 2.0000 so large enough so
most uncertainty does not enter.
Some more examples: It turns out that if a magnetic field is applied to the sample

Now sound waves of low enough \( \omega / q \) undamped

Also spin waves undamped

These effects can be seen optically.