Relativity

The laws of classical mechanics are the same in all "inertial reference frames" at rest or moving with constant velocity.

How defined: Newton's first law holds for objects experiencing no forces, moving in a straight line at constant speed.

Mathematically, \( \vec{F} = m \vec{a} = m \frac{d^2 \vec{r}}{dt^2} \)

\[ \vec{r}' = \vec{r} + \vec{v} t \]

\[ \vec{a}' = \frac{d^2 \vec{r}}{dt^2} = \vec{a} \]

but do forces depend on \( \vec{v} \)? No, even friction is relative velocity of two objects in the frame.

\( E + M \) appears to violate relativity. In rest frame, it changes it produces \( \vec{E} \) but not \( \vec{B} \). Likewise, Lorentz force depends on \( \vec{v} \).
1) To observer moving with wire loop: The loop at rest, there are no magnetic forces on charges in loop but a magnet moves past, changing flux produced electric field (Faraday). Electric force gives $\text{EMF} = -\frac{d\Phi}{dt}$.

2) Observer at rest w.r.t. magnets: The wire loop is in motion, charges feel a magnetic force. If we compute the motional EMF will get $E = -\frac{d\Phi}{dt}$.

\[ F_{mag} = q \mathbf{V} \times \mathbf{B} \rightarrow qV \mathbf{B} \]

\[ Fl = qB \mathbf{V} \mathbf{l} \rightarrow q \frac{d\mathbf{A}}{dt} \]

Same result.

This suggests electromagnetic phenomena also might obey principle of relativity.
Relativity

Of course tied to issue of presence of special

Inertial reference frame at rest with respect to "ether"

the medium whose mechanical vibrations carry light

Michelson-Morley Exp: Speed of light is same in

all directions even though earth moving at 50 km/s

around sun, and sun around galaxy, and galaxy in Universe

LIGO

interferometer

correlates

280 trips of 4 km/amu!
Einstein:

- Laws of physics same in all inertial frames
- Speed of light in vacuum is the same for all inertial observers

\[ V_{AC} = V_{AB} + V_{BC} \]

\[ V_{AC} = \frac{V_{AB} + V_{BC}}{1 + V_{AB}V_{BC}/c^2} \]

Reduces to "common sense" rule when \( V_{AB}, V_{BC} \ll c \)

and when \( V_{BC} = c \)

\[ V_{AC} = \frac{V_{AB} + c}{1 + V_{AB}/c} = c \]

Relativity is governed mathematically by Lorentz transformations.
Before introducing these, consider 3 key phenomena qualitatively.
Sound problem continued

Observer outside rain

Sound going to right moves at $V + V_s$

Sound going to left moves at $V - V_s$

\[ \begin{align*}
\text{pulse traveling to right:} & \quad (V + V_s)t_B = \frac{d}{2} + vt_B \\
\text{pulse traveling to left:} & \quad (V - V_s)t_A = -\frac{d}{2} + vt_A \\
\end{align*} \]

\[ t_B = \frac{d}{2V_s} \]

\[ t_A = \frac{d}{2V_s} \]

Same answer as observer on rain

and simultaneous
Qualitative Discussion of Relativity

* Relativity of Simultaneity

![Diagram showing light emitted from center of a rectangle representing a moving train, with events labeled as simultaneous in one frame but not in another.]

1) Observer inside train: Light reaches A and B at same time

2) Observer on ground: Light reaches A before B

Events which are simultaneous in one frame are not so in another.

Important: Discussion has nothing to do with time it takes for results of event to get to observer. The observer has assistants at A and B with synchronized watches who record arrival of light right at the scene.

What would happen to sound waves?

![Diagram of a freely sliding pole with a strike point at the center.]

* Strike pole at center, so sound wave propagates.

What does observer outside train say?
Time Dilation

Consider light ray leaving bulb going downwards.

Observer on train: \[ \Delta t' = \frac{h}{c} \]

Observer outside train: \[ \Delta t' = \frac{\sqrt{h^2 + v^2 (\Delta t)^2}}{c} \]

Again: interesting to analyze this for sound.

Assumed "h" is same for two observers.

\[ \Delta t = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \Delta t' \]

Moving clocks run slow: \( \Delta t < \Delta t' \)

Evidence: muon lifetime = \( 2 \times 10^{-6} \) sec

Particle accelerators muons moving close to speed of light: lifetime \( \gg 2 \times 10^{-6} \)
Possible contradiction: Who is moving, observer on train or observer on ground? Then if this is not answerable, whose clock runs slow?

Two observers, each with cargo of muons. They move with velocity \( v \) wrt each other. Who has more muons in the future? Answer fixed to question if when in future they compare: Simultaneous does not have same meaning for them, and they need to compare at the same time.
* Lorentz contraction

\[ \Delta x' = \frac{\Delta x}{\sqrt{1 - \frac{v^2}{c^2}}} \]

Observer in train

\[ \Delta t' = \frac{2 \Delta x'}{c} \]

Mirror moving away

Observer on ground

\[ \Delta t_1 = \frac{\Delta x + v \Delta t_1}{c} \]

Light travels extra distance

\[ \Delta t_2 = \frac{\Delta x - v \Delta t_2}{c} \]

Receiver moving towards, light travels less far

\[ \Delta t_1' = \frac{\Delta x}{c - v} \]

\[ \Delta t_2' = \frac{\Delta x'}{c + v} \]

\[ \Delta t = \Delta x \frac{2c}{c^2 - v^2} \]

\[ \Delta t' = \sqrt{1 - \frac{v^2}{c^2}} \Delta t \]

So

\[ \frac{2 \Delta x'}{c} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{2 \Delta x'}{c} \frac{1}{c} \]

\[ \Delta x' = \frac{\Delta x - \Delta x'}{\sqrt{1 - \frac{v^2}{c^2}}} \]

Observer on ground

Moving objects are shortened

\[ \Delta x' = \frac{\Delta x}{\sqrt{1 - \frac{v^2}{c^2}}} \]
Again, amusing to check for Lorentz

contraction via "usual" or "classical"

notion of velocity addition

\[ \Delta x_1 = \Delta x + \frac{v \Delta t_1}{c+v} \]

\[ \Delta t_2 = \frac{\Delta x - v \Delta t_1}{c-v} \]

Speed relative is changed by motion of

Evidently:

\[ c \Delta t_1 = \Delta x \]

\[ c \Delta t_2 = \Delta x \]

agrees perfectly with

\[ \Delta t' = \frac{2 \Delta x'}{c} \]
Ladder in barn paradox, like bundle of moon paradox, has resolution in question of simultaneity:

"In the barn" means both ends of ladder inside at same time, and 2 observers disagree about what "at same time" means.

Lorentz contraction applies only to along direction of motion. Dimensions in length are not contracted.

Wall beside railroad tracks red passenger on train holds paint brush out of window 1 meter above train floor in her frame, observer outside train paint blue line 1 m above ground in her frame.

Which is higher?
Lorentz Transformation (Skip)

Event: something that takes place at specific location \((x,y,z)\) at specific time \(t\)

How are \((x',y',z')\) and \(t'\) in a frame \(S'\) moving wrt frame \(S\) related to \((x,y,z)\) and \(t\)?

From viewpoint of person in \(S\):

\[
x = d + vt \\
x' = x - vt \\
y' = y \\
z' = z \\
t' = t \text{ of course time is same}
\]
But $d \neq x'$  

$d$ is the distance between $O'$ and $A'$ in $S$  

 Whereas $x'$ is distance between $O'$ and $A'$ in $S'$  

 $O'$ and $A'$ are at rest in $S'$, so $x'$ appears contracted in $S'$:  

$$d = \frac{x'}{\gamma} \quad x' = d \gamma$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} > 1$$

$$\frac{1}{\gamma} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} < 1$$

$$x' = \gamma(x - vt) \quad x = \gamma(x' + vt')$$

$$y' = y$$

$$\zeta' = \zeta$$

$$t' = \gamma \left( t - \frac{vx}{c^2} \right) \quad t = \gamma \left( t' + \frac{vx'}{c^2} \right)$$

This follows from using the symmetrical reln

$$x = \gamma(x' + vt')$$

$$= \gamma \left( \gamma(x - vt) + vt' \right)$$

$$= \gamma^2 x - \gamma^2 vt + \gamma vt'$$

$$t' = \gamma t + \frac{1}{\gamma} \left( 1 - \gamma^2 \right) x$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}}$$

$$1 - \gamma^2 = 1 - \frac{1}{1 - \frac{v^2}{c^2}}$$

$$t' = \gamma t + \gamma \frac{vx}{c^2}$$
Can rediscover simultaneity, time dilation, and Lorentz contraction for Lorentz transformation.

Simultaneity: Event A: \( x_A = 0 \) \( t_A = 0 \)

Event B: \( x_B = b \) \( t_B = 0 \)

\[ \begin{align*}
    x'_A &= \gamma (x_A - vt_A) = 0 \\
    t'_A &= \gamma (t_A - \frac{vx_A}{c^2}) = 0 \\
    x'_B &= \gamma (x_B - vt_B) = \gamma b \\
    t'_B &= \gamma (t_B - \frac{vx_B}{c^2}) = -\frac{\gamma vb}{c^2} \quad \text{B occurs before A}
\end{align*} \]"
Einstein's velocity addition rule.
Write Lorentz transformation in matrix form

\[
\begin{align*}
X^0 &= ct \\
X^1 &= x \\
X^2 &= y \\
X^3 &= z
\end{align*}
\]

Lorentz transform matrix

\[
\begin{pmatrix}
X^0 \\
X^1 \\
X^2 \\
X^3
\end{pmatrix} =
\begin{pmatrix}
\gamma & -\gamma \beta & 0 & 0 \\
-\gamma \beta & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
X^0 \\
X^1 \\
X^2 \\
X^3
\end{pmatrix}
\]

Of course reminiscent of rotation eg about z axis

\[
\begin{pmatrix}
x' \\
y' \\
z'
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\]

\[
(q^\mu)' = \sum_{\nu=0}^{3} \Lambda^\mu_\nu q^\nu
\]

Transformation rule for any "4 vector"
Similiarly even more evident

\[
\cosh \Theta = \beta \\
\sinh \Theta = \beta' \gamma
\]

\[
\cosh^2 \Theta - \sinh^2 \Theta = 1
\]

\[
x^2 - \beta^2 y^2 = \delta^2 (1 - \nu^2/c^2) = 1/v
\]

Rationale:
\[
(x^2 + y^2 + z^2)' = (x^2 + y^2 + z^2)
\]

Length of vector invariant

\[
(-a^0 b^0 + a'^1 b^1 + a'^2 b^2 + a'^3 b^3)'
\]

"Contravariant" = \((-a^0 b^0 + a'^1 b^1 + a'^2 b^2 + a'^3 b^3)\)

Keep track of signs by introducing "covariant"

\[
\alpha_\mu = (a_0, a_1, a_2, a_3) = (-a^0, a^1, a^2, a^3)
\]

\[
\alpha_\mu = g_{\mu\nu} a^\nu \\
g_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

"Minkowski Metric"
\[ \sum_{\mu} a_\mu \] is invariant under Lorentz transform.

It is sensible to characterize

\[ a_\mu a^\mu > 0 \quad \text{spacelike} \]
\[ a_\mu a^\mu = 0 \quad \text{lightlike} \]
\[ a_\mu a^\mu < 0 \quad \text{timelike} \]

All observers will agree:

\[ \Delta x^\mu = x^\mu_B - x^\mu_A \]

\[ \Delta x^\mu \] : There is an inertial system where two events occur at some point.

Spacelike: Can find inertial system where events occur at some time (simultaneous).
Relativistic momentum

\[ \vec{p} = \frac{m \vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma m \vec{v} \]

\[ E = cp^0 = mc^2 \frac{\gamma}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma mc^2 \]

\[ p^\mu \text{ is a 4-vector like } x^\mu \text{ obeying} \]

same transformation eqns.

Keep fact: with these definitions energy and momentum are conserved. This rests on exact just as Lorentz transformation rests on Michellon-Morley demonstration that c is same in all reference frames.

\[ p^\mu \cdot \varepsilon = -(p^0)^2 + (p^1)^2 + (p^2)^2 + (p^3)^2 \]

\[ = - \gamma^2 m^2 c^2 + m^2 \frac{v^2}{1 - \frac{v^2}{c^2}} \]

\[ = + \gamma^2 c^2 m^2 \left( -1 + \frac{v^2}{c^2} \right) = -m^2 c^2 \]

\[ = - \frac{1}{\gamma^2} \]

\[ E^2 - c^2 p^2 = m^2 c^4 \]

For a photon (massless particle) \( E = cp \)
Distinguish invariant – same in all inchoate hours

\[ \underline{\text{conserved}} \quad \text{same before and after some process in given frame} \]

\[ M : \text{invariant, not conserved} \]

\[ E : \text{conserved, not invariant} \]

\[ q : \text{conserved and invariant} \]

\[ V : \text{not conserved, not invariant} \]
\[
\frac{3}{5}c \quad \begin{array}{c} \rightarrow \\ m \end{array} \quad \begin{array}{c} \leftarrow \\ m \end{array} \quad \frac{3}{5}c
\]

Conservation of momentum is trivial: zero before and after.

Energy before:
\[
\frac{mc^2}{\sqrt{1 - (\frac{3}{5})^2}} = \frac{5}{4} mc^2
\]
\[
\times 2
\]

Energy after:
\[
M c^2 = \frac{5}{2} mc^2
\]
\[
M = \frac{5}{2} m \quad \text{more than } 2m!
\]

"Final energy" or "inertial energy"
represented as mass of composite object.

\[
\gamma:\quad p_x \sin \phi = p_y \sin \theta
\]
\[
\gamma:\quad p_x \sin \phi = p_y \sin \theta
\]
\[
\sin \phi = \frac{E}{p_c c}
\]
\[ x' = P_x \cos \theta + P_e \cos \phi = \frac{E_0}{c} \]

\[ \frac{E}{c} \cos \theta + P_e \sqrt{1 - \left(\frac{E \sin \phi}{P_e c}\right)^2} = \frac{E_0}{c} \]

\[ \cos^2 \phi = 1 - \sin^2 \phi \]

\[ = 1 - \left(\frac{E}{P_e c}\right)^2 \]

\[ P_e^2 \left(1 - \frac{E^2}{P_e^2 c^2} \sin^2 \theta \right) = \left(\frac{E_0 - E \cos \theta}{c}\right)^2 \]

\[ P_e^2 E^2 = (E - E \cos \theta)^2 + E^2 \sin^2 \theta \]

\[ = E_0^2 - 2E E_0 \cos \theta + E^2 \]

Energy conservation

\[ E_0^2 + m_e c^2 = E + \sqrt{m_e^2 c^2 + P_e^2 c^2} \]

\[ = E + \sqrt{m_e^2 c^4 + E_0^2 - 2E E_0 \cos \theta + E^2} \]

So for low \( E = \frac{1}{(1 - \cos \theta) mc^2 + \sqrt{E_0}} \)

{
\begin{align*}
\text{Using definition} & \quad E = h\nu = \frac{hc}{\lambda} \\
\lambda &= \lambda_0 + \frac{h}{mc} (1 - \cos \theta)
\end{align*}
}
\[(E_0 + mc^2 - E) = \sqrt{m_e c^4 + E_0^2 - 2EE_0 \cos \theta + E^2}
\]

\[
E^2 + m^2 c^4 + E_0^2 + 2E_0 mc^2 - 2E_0 E - 2E_0 mc^2
\]

\[
= m_e c^4 + E_0^2 + E^2 - 2EE_0 \cos \theta
\]

\[
E_0 mc^2 = E\left(E_0 + mc^2 - 2E_0 \cos \theta\right)
\]

\[
E = \frac{E_0 mc^2}{E_0 \left(1 - \cos \theta\right) + mc^2}
\]

\[
= \frac{1}{\frac{1 - \cos \theta}{mc^2} + \frac{1}{E_0}}
\]