We have considered magneto- and electrostatics.

What if $\frac{dB}{dt}$ and/or $\frac{dE}{dt}$ nonzero?

North pole of magnet

\[ \vec{F} = q \vec{v} \times \vec{B} \]

Suppose, however, we considered the equivalent problem of moving the magnet with $+q$ at rest.

Since charges at rest experience only electric forces, we are forced to conclude the moving magnet ($\frac{dB}{dt} \neq 0$) produces an electric field.

We can think about the charge $+q$ as being in a wire and then measuring the potential between the ends of the wire by attaching a voltmeter (more wired).
We know the work done as calculated from the original point of view of pushing against the magnetic force: \[ W = q \mathbf{v} \mathbf{B} \cdot \mathbf{l} \]

so the potential is: \[ \frac{W}{q} = \int \mathbf{E} \cdot d\mathbf{l} = v \mathbf{B} \cdot \mathbf{l} \]

Switching to the viewpoint of the magnet moving, we notice that:

\[ \Phi_B = B \mathbf{l} \times \mathbf{E} \]

\[ \frac{d\Phi_B}{dt} = B \mathbf{E} \cdot d\mathbf{l} \]

In fact, \[ \frac{d\Phi_B}{dt} < 0 \] so

\[ \int \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt} \]

Although derived for a particular geometry, this law is true in general.

\[ E = \frac{1}{4\pi e_0} \frac{q}{r^2} \] for specific geometry, general law.
Rewrite using Gauss' law:

\[-2\frac{\partial \mathbf{B}}{\partial t} = \oint \mathbf{E} \cdot d\mathbf{A}\]

\[-2\frac{\partial}{\partial t} \oint \mathbf{B} \cdot \hat{n} dA = \oint (\nabla \times \mathbf{E}) \cdot \hat{n} dA\]

\[-2\frac{\partial}{\partial t} = \nabla \times \mathbf{E}\]

Previously \( \nabla \times \mathbf{E} = 0\)

\(\mathbf{E} = -\frac{\partial \hat{\phi}}{\partial t}\)

Since

\(\mathbf{B} = \nabla \times \mathbf{A}\)

\(\mathbf{E} = -\hat{\phi} - \frac{\partial \mathbf{A}}{\partial t}\)

Recall one more thing... \(\nabla \times \mathbf{B} = \mu_0 \mathbf{J}\)

\(\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J}\)

\(-\nabla^2 \mathbf{A} + \nabla (\nabla \cdot \mathbf{A}) = \mu_0 \mathbf{J}\)

Choose (Coulomb gauge) \(\nabla \cdot \mathbf{A} = 0\)

\(-\nabla^2 \mathbf{A} = \mu_0 \mathbf{J}\)

Agreed with our magnetostatic identity:

\[\Rightarrow \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(r')}{r} \, d^3 r'\]

point-mass dipole 

\(\Rightarrow \mathbf{B} \) field