**Electrostatic fields in the presence of matter**

We have discussed electric fields of point charges \( q \) and of charge distributions \( p(\mathbf{r}) \), perhaps subject to certain body conditions like the presence of a metal which forces \( \mathbf{E} \) to be \( \mathbf{E} = 0 \) to the surface.

We now turn more generally to the effect of the presence of matter on \( \mathbf{E} \) fields (and, later, \( \mathbf{B} \) fields). In a sense this is not a new topic at all, because matter is just a collection of charges.

However, it is often convenient to isolate the "free charges" \( q \) over which we have experimental control, from the charges in a material which are responding to the free charges.

Indeed, if you think about it, treating the true \( \mathbf{E}, \mathbf{B} \) fields in matter is going to be impossible - vast numbers (10\(^{23}\)) of \( q \), \( \mathbf{E} \) which wildly vary in space and in time. Inevitably we need to average these microscopic fluctuations (unless our expts probes them explicitly, as some do!), this is our motivation for developing models of how matter affects \( \mathbf{E} \) fields.
The simplest model is to treat matter as a "dielectric". We imagine a material

- has no free charge, all charges attached to specific atoms/molecules
- \( \mathbf{E} \) fields produce small displacements from equilibrium
- Effect of \( \mathbf{E} \) can be viewed as displacement of all charge
  of dielectric relative to charge \( \Rightarrow \) material is "polarized"

Physical picture:

\begin{align*}
\text{accumulation of charge} & \quad \Rightarrow \\
\text{induced } \mathbf{E} & \quad \text{inside} \\
\text{opposed } \mathbf{E} & \quad \text{outside}
\end{align*}

Our goal: compute total \( \mathbf{E} \) inside

\[ \text{looks as if } \pm Q \text{ is reduced as in case at left} \]
Is it reasonable to think displacements are small? Can we rip an electron away from a proton?

\[ E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \]

\[ = 9 \times 10^9 \frac{1.6 \times 10^{-19}}{(5 \times 10^{-10})^2} \sim 10^{12} \frac{\text{V}}{\text{m}} \times \frac{\text{N}}{\text{C}} \]

Filed applied? Anyone with lab experience?

"Household experience" 120 V wire is 1 m long \( \sim 10^2 \frac{\text{V}}{\text{m}} \)

Can thermal effects rip an electron from a proton?

Energy \( \sim 13.6 \text{eV} \rightarrow 10^5 \text{ K} \)

1 eV = 12,000 K

But some materials (e.g., semiconductors) bind energy less and can see effects of \( T \) on \# of carriers.

Electric field in lasers

Energy density \( \frac{1}{2} \varepsilon_0 E^2 \)

Multiply by speed \( \frac{1}{2} \varepsilon_0 E^2 c = \frac{\text{Energy}}{\text{area} \times \text{time}} \)

\[ \frac{1}{9\pi\varepsilon_0} = 9 \times 10^9 \quad \varepsilon_0 = 8.85 \times 10^{-12} \]
National Ignition Facility 10^{19} \text{ Watts/m}^2

Silicon power 1 \text{ KW/m}^2 \quad 10^3 = \frac{1}{2} (8.85 \cdot 10^{-12}) E^2 \quad 3 \cdot 10^8

E \approx 10^3 \text{ V/m}

Cell phone? Is it dangerous?
We could get $E$ inside if we knew the size of the accumulated charge at the surface.

$$E_{\text{inside}} = E - \frac{q}{\varepsilon_0}$$

Our basic assumption is $\vec{p} = \gamma \vec{E}$, an applied field.

We will write down some simple models which allow us to compute $\vec{p}$ later, but this is a reasonable assumption. 

Basically, just $F = k \frac{q}{r^2}$ — force on displaced charge.

$$E \vec{q} = k \frac{q}{r^2} = k \frac{P}{\varepsilon_0}$$

So $\vec{p} \propto \vec{E}$. 
It is useful to review the \( \vec{E} \) field and potential \( V \) due to a dipole.

\[
V(r) = \frac{1}{4\pi\varepsilon_0} \left\{ \frac{q_+}{r_+} - \frac{q_-}{r_-} \right\}
\]

\[
r_+^2 = r^2 + d^2 - 2rd\cos\theta
\]

\[
r_-^2 = r^2 + d^2 + 2rd\cos\theta
\]

Expanding,

\[
V(r) = \frac{1}{4\pi\varepsilon_0} \frac{q_d \cos\theta}{r^2}
\]

\[
= \frac{1}{4\pi\varepsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^3}
\]

\[
\hat{p} \cdot \vec{r} = q_d
\]

Also,

\[
\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{\vec{\nabla} [q_d \cos\theta]}{r^3}
\]

For a distribution of dipoles,

\[
V(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\vec{p}(\vec{r}')} \cdot (\vec{r} - \vec{r}') \frac{d^3r'}{1|\vec{r} - \vec{r}'|^3}
\]

\[
= \frac{1}{4\pi\varepsilon_0} \int \vec{p}(\vec{r}') \cdot \vec{\nabla} \left[ \frac{1}{|\vec{r} - \vec{r}'|} \right] d^3r'
\]
\[ V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \left\{ \int \mathbf{P}(\mathbf{r'}) \cdot \frac{d^3 r'}{|\mathbf{r} - \mathbf{r'}|} - \int \mathbf{\hat{r}} \cdot \mathbf{P}(\mathbf{r'}) \cdot d^3 r' \right\} \]

\[ \nabla \cdot \mathbf{P} = \frac{1}{4\pi\varepsilon_0} \left\{ \int \mathbf{P}(\mathbf{r'}) \cdot \mathbf{n'} \cdot d\mathbf{a'} - \int \mathbf{D'} \cdot \mathbf{P}(\mathbf{r'}) \cdot d^3 r' \right\} \]

\[ \mathbf{J}_b = \mathbf{P}(\mathbf{r'}) \cdot \mathbf{n'} \quad \text{the surface charge density} \]

\[ \rho_b = -\nabla' \cdot \mathbf{P}(\mathbf{r'}) \quad \text{the bound charge density} \]

\[ V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \left\{ \int \mathbf{J}_b(\mathbf{r'}) \cdot d\mathbf{a'} + \int \rho_b(\mathbf{r'}) \cdot d^3 r' \right\} \]

Turning to our simple geometry, since \( \nabla \cdot \mathbf{E} = 0 \), and \( \mathbf{P} = \gamma \mathbf{E} \) we have \( \nabla \cdot \mathbf{P} = 0 \), no bound charge density.

Physically:

\[ \mathbf{J}_b = \mathbf{P} \cdot \mathbf{n'} \]

Meanwhile \( \mathbf{J}_b = \mathbf{P} \cdot \mathbf{n'} \)

\[ = -\gamma \mathbf{E} \text{ on left} \quad \text{to finish problem} \]

\[ + \gamma \mathbf{E} \text{ on right} \quad \text{someone needs to tell us material property} \]

\[ \text{namely a value for } \gamma \]
\[ p_{\text{tot}} = p_f + p_b \]

\[ \varepsilon_0 \nabla \cdot \vec{E} = p - \nabla \cdot \vec{p} \]

\[ \nabla \cdot (\varepsilon_0 \vec{E} + \vec{p}) = \rho_f \]

\[ \vec{D}, \text{ the electric displacement} \]

As earlier, one usually writes \( \vec{p} = \varepsilon_0 \chi_e \vec{E} \), electric susceptibility

\[ \vec{D} = \varepsilon_0 (1 + \chi_e) \vec{E} = \varepsilon \vec{E} \]

"dielectric constant" \( \varepsilon / \varepsilon_0 \)

\[ \vec{M} = \chi_m \vec{B} \text{ similarly} \]

\[ \vec{\mu} \text{ magnetic susceptibility} \]
\[ \mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \]

*electric susceptibility*

\[ \sigma_b = \mathbf{P} \cdot \hat{n} = E \]

\[ \mathbf{P} = -\varepsilon_0 \chi_e \mathbf{E} \]

\[ \mathbf{P}_b = -\nabla \cdot \mathbf{P} = -\nabla \cdot (\varepsilon_0 \chi_e \mathbf{E}) = -\varepsilon_0 \chi_e \nabla \cdot \mathbf{E} = -\chi_e \mathbf{P} \]

If constant

\[ \mathbf{P} = P_f \]

\[ \mathbf{P}_b = -\chi_e \frac{P_f}{1 + \chi_e} \]

If \( P_f = 0 \) so \( 100 \) is \( \mathbf{P}_b \)!

\[ \mathbf{P} = \frac{P_f}{1 + \chi_e} \]

*Polarization cloud screens + q*

\[ \mathbf{P} = \frac{P_f}{1 + \chi_e} \]

Total change is reduced from \( P_f \) density
Boundary conditions since \( \vec{D} \cdot \vec{n} = p \) free

The discontinuity of \( \vec{D} \) is proportional to free charge density \( \vec{p} \) on surface.

\[ D_{2n} A - D_{1n} A = \sigma_{\text{free}} A \]

\[ E_2 \cdot \vec{d}l + E_1 (-\vec{d}l) = 0 \]

\[ E_{2t} = E_{1t} \]
\[ \nabla \cdot \vec{E} = \frac{\rho}{\epsilon} \]
\[ \vec{E} = -\nabla V \]
\[ \nabla^2 V = -\frac{\rho}{\epsilon} \]

If no free charge density still have Laplace’s eqn
\[ \nabla^2 V = 0 \]

So all our old techniques still work, we just need to make sure we adapt them to correct boundary conditions.

\[ \varepsilon_2 \quad \varepsilon_1 \]

Point charge near interface of two different dielectric regions (often one of \( \varepsilon_1, \varepsilon_2 \) is equal to vacuum)

\[ E_x \bigg|_{z=0^+} = E_x \bigg|_{z=0^-} \]
\[ E_y \bigg|_{z=0^+} = E_y \bigg|_{z=0^-} \]
\[ \varepsilon_1 E_z \bigg|_{z=0^+} = E_z \bigg|_{z=0^-} + \int_{-D} \nabla \cdot E \]

\[ E \times \vec{E} = 0 \text{ still holds} \]
Michael asks (I think):

"What about expectation that V will depend on both $\varepsilon_1$ and $\varepsilon_2$ because V represents work done bringing test charge in from $\infty$ and the path lies partly in region 1 and partly in region 2?"

Let's do the calculation:

$V_1 - V_\infty = -\int_\infty^{q_1} \vec{E} \cdot d\vec{r} = \frac{q}{4\pi\varepsilon_0 R_1} + \frac{q}{4\pi\varepsilon_1 R_2}$

$V_2 - V_\infty = V_2 - V_1 + V_1 - V_\infty = \frac{q}{4\pi\varepsilon_0 R_1} + \frac{q}{4\pi\varepsilon_1 R_2}$
Connection to QM

Hydrogen atom: \( a_0 = \frac{4\pi\varepsilon_0 k^2}{M_e e^2} \)

100: \( \frac{1}{i} \frac{1}{a_0^{3/2}} \frac{1}{r/a_0} e^{-r/a_0} \)

200: \( \frac{1}{4\sqrt{2\pi}} \frac{1}{a_0^{3/2}} \frac{1}{2 - r/a_0} e^{-r/a_0} \)

210: \( \frac{1}{4\sqrt{2\pi}} \frac{1}{a_0^{3/2}} \frac{r}{a_0} e^{-r/a_0} \cos \theta \)

21±1: \( \frac{1}{8\sqrt{\pi}} \frac{1}{a_0^{3/2}} \frac{r}{a_0} e^{r/a_0} \sin \theta e^{\pm i\phi} \)

\( R_{n\ell m}(r) Y_{\ell m}(\theta, \phi) \) familiar

\( E_{n\ell m} = \frac{E^2}{2a_0} \frac{1}{n^2} \)

\( P(\ell, m) \) moment of \( \ell, m \)

200, 210, 21±1 all equally likely

P spherical symmetry

apply \( E = E_o e^2 \) angular structure

\[ P(\ell, m, \Theta) \sim 1 \]

\[ P(210) \sim \cos^2 \Theta \]

\[ P(21\pm1) \sim \sin \Theta \]

Equally likely...
Linear combination!

\[ |200\rangle - |210\rangle \sim 1 - \cos \theta \quad \text{peaked at } \theta = \pi \]

\[ |200\rangle + |210\rangle \sim 1 + \cos \theta \quad \text{peaked at } \theta = 0 \]

**Q.** Which is lower for electron? & A. negatively charged

should move to \( z < 0 \)

(opposite to \( E \))

hence \( \theta = \pi \)

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**Q.** Did you do this problem before? How?

**A.** Yes, Stark effect e.g. by perturbation theory

\[ H^{(1)} = -eEz \]

\[ E^{(1)}_{\text{rem}} = -eE_0 \langle n\ell m | z | n\ell m \rangle = 0 \]

Conclude no first order shift

\[ E_0 \quad \text{simdcasee} \]

\[ 210 \quad \text{simdcasee} \]

\[ 21-1 \quad \text{simdcasee} \]

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**Q.** Is that right?!

**A.** No! If states degenerate need to diagonalize matrix

\[
\begin{pmatrix}
200 & 210 & 211 & 21-1 \\
200 & 0 & x & 0 & 0 \\
210 & x & 0 & 0 & 0 \\
211 & 0 & 0 & 0 & 0 \\
21-1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
The perturbed ground state is a linear combination of $|210\rangle$ as we intuitively expected.

Details: $|200\rangle \equiv |210\rangle$ angular part

$$\sim \int \frac{d\theta \sin \theta}{3} \cos \theta \frac{1}{\rho^2} \cos \theta \sim \frac{-1}{3} \cos^3 \theta \bigg|_0^\pi \neq 0$$

$$\sim \frac{1}{3} \times (1 \cdot 2 \cdot 211) - \frac{1}{3} (-1 \cdot 1 \cdot 1)$$

$|200\rangle \equiv |211\rangle$

$$\sim \int \frac{d\theta \sin \theta}{3} \cos \theta \sin \theta \frac{1}{\rho^2} e^{i\phi} \bigg|_0^\pi \sim \frac{1}{3} \times (1 \cdot 2 \cdot 211) \text{ vanishes also!}$$

$$\sim \sin^3 \theta \bigg|_0^\pi = 0$$

etc.
Q. Any ideas for method?

A. Images < Rules of game:
  can monkey around with \( g(\mathbf{r}) \) outside
  region of \( V \) to heart's delight.

a) \( z > 0 \)

\[
\begin{align*}
eg_2 & \quad \neg_1 \\
\mathcal{R}_2 & \quad \mathcal{R}_1 \\
g & \quad d \\
d & \quad q
\end{align*}
\]

Below: \( g \) is distance \( \sqrt{x^2+y^2} \)

b) \( z < 0 \)

\[
\begin{align*}
eg_2 & \quad \neg_1 \\
\mathcal{R}_2 & \quad \mathcal{R}_1 \\
g & \quad d \\
d & \quad q''
\end{align*}
\]

Q. Why are we forced to use "real" \( q \) for \( z > 0 \)
but allowed fictitious \( q'' \) for \( z < 0 \)

\[
V = \frac{q}{4\pi \varepsilon_1 R_1} + \frac{q'}{4\pi \varepsilon_1 R_2} \\
V = \frac{q''}{4\pi \varepsilon_2 R_1}
\]

\[
\text{tangential} \quad E = -\frac{1}{4\pi \varepsilon} \left( \frac{\frac{q}{R_1} + \frac{q'}{R_2}}{2} \right) = -\frac{1}{4\pi \varepsilon} \left( \frac{q''}{R_1} \right) \bigg|_{z=0}
\]

\[
\frac{q''}{(p^2+d^2)^{3/2}} = \frac{1}{\varepsilon_2} \frac{q''}{(p^2+d^2)^{3/2}}
\]

\[
\frac{1}{\varepsilon_1} \left( g + g' \right) = \frac{1}{\varepsilon_2} q''
\]

NB. This also makes the \( z > 0 \) and \( z < 0 \) forms for \( V \) equal at \( R_1 = R_2 \), i.e., along the dividing line where 2 regions touch.
\[-\frac{1}{4\pi} \frac{\partial^2}{\partial z^2} \left( \frac{q}{R_1} + \frac{q'}{R_2} \right) \bigg|_{z=0} = -\frac{1}{4\pi \varepsilon_0} \frac{\partial}{\partial z} \frac{q''}{R_1} \bigg|_{z=0} \]

\[\frac{q - q'}{\sqrt{p^2 + d^2}} = \frac{q''}{\sqrt{p^2 + d^2}}\]

\[q - q' = q''\]

\[q'' = \frac{2\varepsilon_2}{\varepsilon_2 + \varepsilon_1} q\]

Surface charge density: \( \sigma_b = \hat{P} \cdot \hat{n} \)

\[\sigma_b = \hat{P}_1 \cdot \hat{n} \varepsilon_2 + \hat{P}_2 \cdot \hat{n} = -\hat{P}_1 \varepsilon_2 + \hat{P}_2 \varepsilon_2\]

\[\hat{P}_2 = (\varepsilon - \varepsilon_0) E_2\]

\[\sigma_b = (\varepsilon_1 - \varepsilon_0) \frac{1}{4\pi \varepsilon_1} \frac{d(q - q')}{\sqrt{p^2 + d^2}} - (\varepsilon_2 - \varepsilon_0) \frac{1}{4\pi \varepsilon_2} \frac{q'' d}{\sqrt{p^2 + d^2}}\]

\[= -\frac{q}{2\pi} \frac{\varepsilon_0 (\varepsilon_2 - \varepsilon_1)}{\varepsilon_1 (\varepsilon_2 + \varepsilon_1)} \frac{d}{\sqrt{p^2 + d^2}}\]

\( \Delta \) vanishes when \( \varepsilon_1 = \varepsilon_2 \) as expected!

Q: But shouldn't it be symmetric under \( \varepsilon_1 \neq \varepsilon_2 \)? Why \( \varepsilon_1 \) special?!
EM

Any limit you can check

Metal $\varepsilon_2 \rightarrow \infty$

$$q' = -\left(\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1}\right)q \rightarrow -q \quad \forall V$$

Tangential

$$\frac{\varepsilon_1 (q + q')}{\varepsilon_1} = \varepsilon_2 q''$$

both vanish

But

$$q'' = \left(\frac{2\varepsilon_2}{\varepsilon_2 + \varepsilon_1}\right)q \rightarrow 2q$$

No problem $V$ for $\varepsilon_2 < 0$ is $q''/4\pi\varepsilon_2 R_1$

So value of $q''$ irrelevant $V \rightarrow 0$

since $\varepsilon_2 \rightarrow -\infty$
A classic problem: dielectric sphere in uniform E field

\[ r > a \quad V(r) = \sum [B_{2l} r^l + c_{2l} r^{-(l+1)}] P_l(\cos \theta) \]

\[ \nabla \phi = -E_0 z \]

\[ \phi = -E_0 r \cos \theta \]

\[ P_l(\cos \theta) \]

\[ r < a \quad V(r) = \sum A_{2l} r^l P_l(\cos \theta) \]

tangent to E

\[ -\frac{1}{a} \frac{\partial V}{\partial \theta} \bigg|_{r=a} = -\frac{1}{a} \frac{\partial V}{\partial \theta} \bigg|_{r=a} \]

Usual argument: coefficients of \( P_l(\cos \theta) \) on either side must match term-by-term. The same is true even if \( \frac{\partial}{\partial \theta} P_l(\cos \theta) \)

\[ \lambda = 1 \quad A_1 = -E_0 + C_1/a^3 \]

\[ \lambda \neq 1 \quad A_2 = C_2/a^{2l+1} \]
\[- \frac{\partial \mathbf{V}}{\partial r} \bigg|_{r=a} = - \varepsilon_0 \frac{\partial \mathbf{V}}{\partial r} \bigg|_{r=a} = \varepsilon_0 \sum_k \mathbf{A}_k \rho_0 \frac{r-1}{r} \mathbf{p}_k (\cos \theta) \]

\[= \varepsilon_0 \mathbf{E}_o \cos \theta + \varepsilon_0 \sum_k (k+1) \frac{C_k}{a^{2k+1}} \mathbf{p}_k (\cos \theta) \]

\[l=1 \quad - \varepsilon/\varepsilon_0 \quad A_1 = -E_o + 2c_1/a^3 \]

\[l \neq 1 \quad - \varepsilon/\varepsilon_0 \quad A_e = (l+1) C_e/a^{2l+1} \]

Obviously \( \mathbf{A}_e = 0 \quad l \neq 1 \)

and \(- \varepsilon/\varepsilon_0 \left\{ -E_o + c_1/a^3 \right\} \right\} = -E_o + 2c_1/a^3 \]

\[c_1 = \left( \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right) E_o a^3 \]

\[A_1 = c_1/a^3 - E_o = -3 \left( \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right) E_o \]

Finally \(- \mathbf{E}, \mathbf{P}, \mathbf{D} \text{ constant inside sphere} \)

\[ \left\{ \begin{array}{ll}
   r < a & V_r = -3 \left( \frac{\varepsilon/\varepsilon_0 + 2}{\varepsilon/\varepsilon_0 - 1} \right) E_o r \cos \theta \\
   r > a & V_r = -E_o r \cos \theta + \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} E_o a^3 \cos \theta \end{array} \right. \]
Notice at $r=a$ 

\[ V > \frac{E_0 a \cos \theta}{\varepsilon \varepsilon_0 + 2} \left\{ -1 + \frac{\varepsilon_e - 1}{\varepsilon_e + 2} \right\} \]

\[ -\frac{\varepsilon_e - 2}{\varepsilon_e + 2} = \frac{3}{\varepsilon \varepsilon_0 + 2} \]

So $V_{r=r_a} = \frac{E_0}{\varepsilon \varepsilon_0 + 2}$

For $\varepsilon = \varepsilon_0$, $E_0$ is uniform everywhere.

**Dielectric sphere produces dipole moment**

\[ V = \frac{1}{4\pi \varepsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^3} \]

\[ \Rightarrow \quad \vec{p} = 4\pi \varepsilon_0 \frac{\varepsilon_0 - 1}{\varepsilon_0 + 2} E_0 a^3 \hat{r} \]

**Alternatively, can get**

\[ \vec{p} = (\varepsilon - \varepsilon_0) \vec{E} \text{ inside sphere} \]

\[ \vec{D} - \varepsilon_0 \vec{E} = \frac{3}{(\varepsilon_0 + 2)} E_0 \hat{r} \]

And then

\[ \vec{p} = \frac{4}{3} \pi a^3 \vec{P} = \frac{4\pi \varepsilon_0}{\varepsilon_0 + 2} \vec{E}_0 a^3 \hat{r} \]

Uniform $E_0 \Rightarrow$ constant $\vec{P}$ inside sphere $\Rightarrow \nabla \cdot \vec{P} = 0$

**Surface charge density**

\[ \sigma_b = \frac{\vec{p} \cdot \hat{n}}{\hat{n}} = \frac{\vec{P} \cdot \hat{n}}{\hat{n}} \]

\[ = 3\varepsilon_0 \frac{\varepsilon_0 - 1}{\varepsilon_0 + 2} E_0 \cos \theta \]
\[ \nabla \cdot \mathbf{E} = \frac{p}{\varepsilon_0} = \frac{(p_f + p_b)}{\varepsilon_0} \]

Lines of \( \mathbf{E} \) terminate on \( p_b \) and \( p_f \)

\[ \nabla \cdot \mathbf{P} = -p_b \]

Lines of \( \mathbf{P} \) terminate on \( p_b \)

\[ \nabla \cdot \mathbf{D} = p_f \]

Lines of \( \mathbf{D} \) terminate on \( p_f \)

This is pictorial way of envisioning divergence theorem!

\[ \Phi_E = \int \mathbf{E} \cdot \hat{n} \, dA = \int \nabla \cdot \mathbf{E} \, dV = \frac{1}{\varepsilon_0} \int p \, dV = \frac{q}{\varepsilon_0} \]

Flux of \( \mathbf{E} \) through surface

Monopoles

Dipole
For our dielectric sphere problem:

\[ r < a \quad V_1 = -\frac{3}{\epsilon_0 + 2} E_0 r \cos \theta \]

\[ r > a \quad V_2 = -E_0 r \cos \theta + \frac{\epsilon_0 - 1}{\epsilon_0 + 2} E_0 \frac{a^2}{r^2} \cos \theta \]

Inside sphere \( \mathbf{\dot{E}} = -\nabla V = -\frac{3}{\epsilon_0 + 2} E_0 \mathbf{\hat{z}} \)

and \( \mathbf{\hat{P}} = (\epsilon - \epsilon_0) \mathbf{\dot{E}} \)

\( \mathbf{\hat{P}} = \epsilon \mathbf{\dot{E}} \)

Outside sphere \( \mathbf{\hat{P}} = (\epsilon_0 - \epsilon_0) \mathbf{\dot{E}} = 0 \)

So can draw \( \mathbf{\hat{P}} \)

- Lines start and end on \( \mathcal{S}_b \)
- Why do they go from \( \odot \) to \( \bigcirc \)?
- \( \mathbf{\hat{P}} \cdot \mathbf{\hat{P}} = -P_b \)

Dipole field outside
lines of $\mathbf{E}$ 
\[ V_\phi = -\frac{3}{\epsilon_\phi + 2} \mathbf{E}_0 \cos \theta \]

\[ E_z = E_x = -\frac{1}{r} \partial V_\phi / \partial \theta = -\frac{3}{\epsilon_\phi + 2} \mathbf{E}_0 \sin \theta \]

\[ E_z = -\frac{3}{\epsilon_\phi + 2} \mathbf{E}_0 \cos \theta \]

$D_n$ is continuous so 
\[ \epsilon E^2 = \epsilon_0 E_\phi \]

\[ E_\phi = \frac{\epsilon E_z}{\epsilon_0} \]

It has kink at dielectric surface

since its normal component is reduced abruptly on entering the dielectric.

$D_n$ is continuous so

\[ \epsilon E^2 = \epsilon_0 E_\phi \]

$E_\phi = \frac{\epsilon E_z}{\epsilon_0} \]

Normal component lengthened
a little so not in $z$ direction anymore

[tilted towards $z$ will lengthen $E_n$]
Polarization of single atom

Stark effect perturbation \( H^{\text{Stark}} = -\epsilon E z \)

\[ \Rightarrow \text{shifted wave functions with } \langle z \rangle \neq 0 \]

\[ \Rightarrow \text{polarization } \mathbf{P} = -\epsilon \langle z^2 \rangle \hat{z} \]

\[ \hat{z} \text{ is some number times } E \]

\[ \mathbf{D} = \text{this what we add to } E_0 \mathbf{E} \text{ to get } \mathbf{D} \]

\[ \text{A: Dimensionally wrong} \]

\[ [6\epsilon E] = \frac{\sigma}{L^2} \quad [\mathbf{P}] = \sigma \mathbf{L} \]

Missing \( \frac{1}{L^3} \). What is it?

Physically, the density of atoms must be relevant

\[ \mathbf{P} = -\epsilon n \langle z^2 \rangle \hat{z} \]

Factors of \( n \) are the simplest way "CM physics" enters.

Can get \( \epsilon \), dielectric function of a cloud of \( H \) atoms

by doing the Phys 215 calculation of \( \langle z^2 \rangle \).
Such pictures are exceedingly crude since they treat solids as collections of independent atoms, yet we know crystal structure, electrons break free to form a metal, etc.

Still n plays fundamental role.

Consider a metal as a box of e\textsuperscript{-} which are totally free of their ions.

\[ \psi(x, y, z) = -\frac{1}{\sqrt{V}} \sin \frac{x}{L} \cdot \sin \frac{y}{L} \cdot \sin \frac{z}{L} \]

\[ \psi(x, y, z) = \frac{8}{V} \sin \frac{x}{L} \cdot \sin \frac{y}{L} \cdot \sin \frac{z}{L} \]

\[ e^{i \hbar \omega \tau} - e^{-i \hbar \omega \tau} \]

usual plane wave states which have same energy.
Pauli principle: can only put 2e\(^-\) in any state \((k_x, k_y, k_z)\).

Given \(N\) electrons must fill up states

\[
N = \sum_{\text{occupied}}^\infty \frac{1}{\sqrt{2} \pi^{3/2} k_{\text{max}}^{3/2}} \int d^3k \frac{1}{2}
\]

\[
\frac{N}{V} = \frac{2}{8\pi^3} \frac{4}{3} \pi k_F^3
\]

\[
k_F = \left(3\pi^2 n_F\right)^{1/3}
\]

Our friend \(n_F\) is relevant to metals too!

H\(^{(1)}\) is tricky if many e\(^-\) around because

H\(^{(1)}\) might try to push e\(^-\) into a state already occupied by another e\(^-\): "Pauli blocking."

Lindhard dielectric function. Apply \(\vec{E}\) which has \(\vec{k}\) and \(\omega\) dependence.