PHYSICS 200C, SPRING 2017
ELECTRICITY AND MAGNETISM
Midterm Exam

Instructions: Do one of problems 1 or 2; and one of problems 3 or 4.

[1.] A point charge \( Q \) is situated at position \( (x, y, z) = (0, 0, d) \). The dielectric constant is \( \varepsilon_1 \) for \( z > 0 \) and \( \varepsilon_2 \) for \( z < 0 \). Compute the electrostatic potential at all points in space.

[2.] A dielectric sphere is placed in a uniform electric field. Compute the potential everywhere in space, and the electric field and polarization vectors inside the sphere.

\[
\text{For } V \text{ at } z > 0 \text{ insert image charge } q' \text{ at } (0, 0, -d)
\]

\[
\text{For } V \text{ at } z < 0 \text{ use charge } q'' \text{ at } (0, 0, +d)
\]

\[
V > 0 \quad V = \frac{q}{4\pi \varepsilon_1 R_1} + \frac{q'}{4\pi \varepsilon_1 R_2}
\]

\[
V < 0 \quad V = \frac{q''}{4\pi \varepsilon_2 R_1}
\]

\[
\text{tangential component of } \vec{E} \text{ is continuous at interface } z = 0
\]

\[
\left. \frac{-1}{4\pi \varepsilon_1} \frac{\partial}{\partial z} \left( \frac{q}{R_1} + \frac{q'}{R_2} \right) \right|_{z=0} = \left. \frac{-1}{4\pi \varepsilon_2} \frac{\partial}{\partial z} \frac{q''}{R_1} \right|_{z=0}
\]

\[
\frac{q}{\sqrt{x^2+y^2}}
\]

\[
\frac{1}{\varepsilon_1} \left( \frac{q}{R_1} + \frac{q'}{R_2} \right) = \left( \frac{q''}{R_1} \right)
\]

\[
\text{normal component of } \vec{D} \text{ is continuous at interface } z = 0
\]

\[
\left. \frac{-1}{4\pi \varepsilon_1} \frac{\partial}{\partial z} \left( \frac{q}{R_1} + \frac{q'}{R_2} \right) \right|_{z=0} = \left. \frac{-1}{4\pi \varepsilon_2} \frac{\partial}{\partial z} \frac{q''}{R_1} \right|_{z=0}
\]

\[
\frac{q \cdot d - q' \cdot d}{(p^2 + d^2)^{3/2}} = \frac{q'' \cdot d}{(p^2 + d^2)^{3/2}}
\]

\[
q' = -\left( \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_2 + \varepsilon_1} \right) q \quad ; \quad q'' = \left( \frac{2 \varepsilon_2}{\varepsilon_2 + \varepsilon_1} \right) q
\]
An extension of this problem is to get the bound charge density \( \sigma_b = 9_{12} - 9_{22} \) at \( z = 0 \) interface.

In general, with assumed symmetry, the solutions to Laplace's Eqn \( \nabla^2 V = 0 \):

\[
V(r) = \sum \left[ B_e r^l + C_e r^{-(l+1)} \right] P_l (\cos \theta)
\]

If we use these for \( r > 0 \) we have all \( B_e = 0 \) except

since \( -\nabla^2 V = E_0 \hat{z} \) at \( r = \infty \) we have \( V = -E_0 z = -E_0 \cos \theta \) as \( r \to \infty \)

\[
V_j(r) = -E_0 \cos \theta + \sum C_e r^{-(l+1)} P_l (\cos \theta)
\]

For \( r < 0 \)

\[
V_j(r) = \sum A_e r^l P_l (\cos \theta)
\]

to avoid divergences at \( r = 0 \).

Now use boundary conditions on sphere surface.

Tangential \( \mathbf{E} \):

\[
-\frac{1}{r} \frac{\partial V_j}{\partial \theta} \bigg|_{r=a} = -\frac{1}{a} \frac{\partial V}{\partial \theta} \bigg|_{r=a}
\]

The coefficients \( A_e \) or \( \frac{\partial}{\partial \theta} P_l (\cos \theta) \) must match term-by-term.

\( l = 1 \)

\( A_1 = -E_0 + C_1/a^3 \)

\( l \neq 1 \)

\( A_l = C_l/a^{2l+1} \)

Similarly for normal \( \mathbf{D} \):

\[
-\varepsilon \frac{\partial V_j}{\partial r} \bigg|_{r=a} = -\varepsilon_0 \frac{\partial V}{\partial r} \bigg|_{r=a}
\]

\( l = 1 \)

\( -\varepsilon A_1 = +\varepsilon_0 \left( E_0 + 2C_1/a^3 \right) \)

\( l \neq 1 \)

\( -\varepsilon l A_l = \varepsilon_0 \left( l+1 \right) C_l/a^{2l+1} \)

For \( l \neq 1 \) the \( A_l = C_l = 0 \) are homogeneous \( \Rightarrow \) \( A_1, C_1 \) are nonvanishing.

\( A_1 = \left( \frac{\varepsilon_0 - 1}{\varepsilon_0 + 2} \right) E_0 \)

\( C_1 = \left( \frac{\varepsilon_0 - 1}{\varepsilon_0 + 2} \right) E_0 \)

Inside sphere:

\[
V_j = -\frac{3}{(\varepsilon_0 + 2)} E_0 \cos \theta \frac{1}{a^2}
\]

\( \mathbf{E} = -\nabla V_j = -\frac{3}{(\varepsilon_0 + 2)} E_0 \hat{z} \)

\( \mathbf{D} \) and \( \mathbf{P} \) are also constant within sphere.

\( \mathbf{D} = \varepsilon \mathbf{E} \)

\( \mathbf{P} = (\varepsilon - \varepsilon_0) \mathbf{E} \)
[3.] Starting from the Maxwell Equations, show that the scalar and vector potentials obey separate wave equations when working in the Lorentz gauge.

-OR-

[4.] Derive the Greens function for the wave equation.

\[ \nabla \times \vec{B} = \mu_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \varepsilon_0 \frac{\partial \vec{A}}{\partial t} \]

\[ \nabla \times (\nabla \times \vec{A}) = -\nabla^2 \vec{A} + \nabla (\nabla \cdot \vec{A}) \]

Thus

\[ -\nabla^2 \vec{A} + \nabla (\nabla \cdot \vec{A}) = \mu_0 \frac{\partial \vec{E}}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \frac{1}{c^2} \nabla \cdot \nabla \vec{A} \]

\[ \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = \nabla \cdot \nabla \vec{A} + \frac{1}{c^2} \frac{\partial \vec{A}}{\partial t} \]

Vanishes in Lorentz gauge

\[ \vec{A} \text{ obeys wave eqn with } \mu_0 \frac{\partial \vec{E}}{\partial t} \text{ as source} \]

b) \( \nabla \cdot \vec{E} = \rho/\varepsilon_0 \)

\[ \nabla \cdot (-\nabla \vec{A} - \frac{\partial \vec{A}}{\partial t}) = \rho/\varepsilon_0 \]

\[ \nabla^2 \vec{A} + \frac{\partial}{\partial t} (\nabla \cdot \vec{A}) = -\frac{\rho}{\varepsilon_0} \]

\[ -\frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \text{ in Lorentz gauge} \]

\[ \vec{E} \text{ obeys wave eqn with } +\rho/\varepsilon_0 \text{ as source} \]

4.

We seek \( G(\vec{r},t) \) obeying

\[ (\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) G(\vec{r},t) = \delta^3(\vec{r}) \delta(t) \]

Fourier transform to w space

\[ G(\vec{r},t) = \int_{-\infty}^{\infty} dw \frac{1}{2\pi} e^{-i\vec{w} \cdot \vec{r}} G(\vec{w},t) \]

and notice \( \delta(t) = \int_{-\infty}^{\infty} \frac{dw}{2\pi} e^{-i\omega t} \)

so that (defining \( k = \omega/c \))

\[ (\nabla^2 + k^2) G(\vec{r},w) = \delta^3(\vec{r}) \]

At \( w = k = 0 \) we know the soln is \( \hat{G}(\vec{r},0) = 1/|\vec{r}| \)

If we look for a soln which only depends on \( r = |\vec{r}| \) and not \( \theta, \phi \) even when \( w \neq 0 \) we can simplify the \( \nabla^2 \)
operator...
\[ \nabla^2 \left( \frac{1}{r} \frac{d^2}{dr^2} r \tilde{g} + k^2 \tilde{g} \right) = \delta^3(\tilde{r}) \]

\[ \text{if } n \neq 0, \text{ if } \delta^3(\tilde{r}) = 0 \]

Multiply by \( r \) and note \( r \delta^3(\tilde{r}) = 0 \)

\[ \frac{d}{dr} \left( r \tilde{g} \right) = -k^2 r \tilde{g} \]

\[ \Rightarrow \quad r \tilde{g} = e^{\pm ikr} \]

Hence \( \tilde{g}^\pm (r, \omega) = \frac{1}{r} e^{\pm ikr} \quad k = \omega/c \)

Can now go back to time domain

\[ \tilde{g}^\pm (r, t) = \frac{1}{r} \int \frac{d\omega}{2\pi} e^{-i\omega t} e^{\pm i\omega/c} \]

\[ = \frac{1}{r} \delta(t \mp \gamma/c) \]

Retarded time...

For completeness... you can decide which \( \tilde{g}^\pm \)
obeys causality. We will want

\[ \Phi(r, t) = \int d^3 r' dt' e^{i\frac{kr'c}{4\pi \epsilon_0}} \frac{1}{r-r', t-t'+ \frac{|r-r'|}{c}} \delta(t-t'+ \frac{|r-r'|}{c}) \]

we expect potential at time \( t \) to depend on
change density at \( t' = t - \frac{|r-r'|}{c} \)

A later time, \( t' \)

So \( \Phi \) is \( \tilde{g}^+ \) that obeys causality.