[1.] Compute the potential $V(r, \theta, \phi)$ due to a thin ring of charge $Q$ and radius $R$. Make a convenient choice of origin and orientation of your axes, and assume $r > R$. Interpret the term which falls off most slowly with $1/r$. How does your calculation change for $r < R$?

[2.] A crude model of the H$_2$ molecule is that the electrons form a spherical cloud of charge of radius $a$ and the two protons are point charges inside this sphere. Find the equilibrium proton positions.

[3.] One can verify by explicit integration that the functions sin $(n\pi x/L)$ and cos $(n\pi x/L)$, where $n = 1, 2, 3, \ldots$, are orthogonal and complete on the interval $x \in [0, L]$. This is the basis of Fourier expansion. Similarly, given the specific functional forms of the Legendre polynomials, they can be shown to also to be orthogonal and complete. Write a few sentences describing what more general principle might lie behind the idea of complete sets of functions. Can you make an analogy with the eigenvectors of a specific class of matrices?

[4.] Use the generating function for the Legendre polynomials

$$g(x, t) = \frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_n t^n P_n(x)$$

to prove $P_n(1) = 1$ for all $n$. What can you say about $P_n(0)$?

Do only one of problems [5] or [6] below. In either case, solve the problem completely from scratch, i.e. starting from the appropriate partial differential equation, making a suitable guess at the form of the solution, etc.

[5.] Solve for the potential $V(x, y)$ inside a rectangular box of dimensions $0 < x < b$ and $0 < y < h$ given the boundary conditions $V(x, y = 0) = 0$; $V(x = 0, y) = 0$; $V(x = b, y) = 0$ and $V(x, y = h) = f(x)$ where $f(x)$ is an arbitrary function which vanishes at $x = 0$ and $x = b$. There is no charge inside the box. Identify the Green’s function which arises in your solution.

[6.] Solve for the potential $V(x, y)$ in the upper half-plane $y > 0$ if you are given the potential $V(x, y = 0) = f(x)$ along the $x$ axis. There are no charges present. Suppose $f(x) = V_0$ is constant. What do you get for the potential $V(x, y)$? Identify the Green’s function which arises in your solution.

Potentially Useful Identity:

$$(1 + u)^n = 1 + nu + \frac{n(n-1)}{2} u^2 + \frac{n(n-1)(n-2)}{6} u^3 \ldots$$
We know

\[ V(r, \theta) = \sum_{\ell=0}^{\infty} \left( a_{\ell} r^{\ell} + b_{\ell} r^{\ell+1} \right) P_{\ell}(\cos \phi) \]

for \( \nabla^2 V = 0 \) with azimuthal symmetry.

We can compute \( V(r, \theta = 0) \) (along z-axis).

Since all points of ring are equidistant from such points

\[ V(r, \theta = 0) = \frac{Q}{4\pi\epsilon_0} \left( \frac{1}{r^2 + R^2} \right)^{\frac{1}{2}} \]

assuming \( r > R \)

\[ = \frac{Q}{4\pi\epsilon_0} \left( 1 + \frac{R^2}{r^2} \right)^{\frac{1}{2}} \]

\[ = \frac{Q}{4\pi\epsilon_0} r \left[ 1 - \frac{1}{2} \frac{R^2}{r^2} + \frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \left( \frac{R^2}{r^2} \right)^2 + \ldots \right] \]

\[ V(r, \theta = 0) = \frac{Q}{4\pi\epsilon_0} \left\{ 1 - \frac{1}{2} \frac{R^2}{r^2} + \frac{3}{8} \frac{R^4}{r^4} + \ldots \right\} \]
From Gauss' law, the electric field inside a sphere of constant charge density $p$ is

$$\Phi_E = \frac{Q}{\varepsilon_0}$$

$$4\pi r^2 \varepsilon_0 E = \frac{1}{\varepsilon_0} \left( \frac{4}{3} \pi r^3 \right) p$$

In our case $p = \frac{Q}{4\frac{1}{3} \pi a^3}$

Putting this together,

$$|E| = \frac{pr}{\varepsilon_0} = \frac{Q}{4\pi \varepsilon_0 a^3} r = \frac{e}{2\pi \varepsilon_0 a^3} r$$

The associated force $\vec{F} = \vec{E} \vec{q} = -\frac{e^2}{2\pi \varepsilon_0 a^3} r \hat{r}$

points radially inward (the - charge of e cloud pulls + proton towards origin).

The proton is also repelled by its partner, a radially outward force,

$$\vec{F}_a = \frac{e^2}{4\pi \varepsilon_0 (2r)^2} \hat{r}$$

$$\vec{F}_{10} = \frac{e^2}{a^3} \left\{ -\frac{2r}{a^3} + \frac{1}{4r^2} \right\} = 0 \text{ at equilibrium}$$

$$8r^3 = a^3 \quad \therefore r = \frac{a}{2}$$
NOTE: I expect a much shorter answer from you, which, however, hits some of key points.

The collection of all (complex valued) functions \( f(x) \) (infinite dimension!)
can be regarded as a vector space. The inner product
of two such "vectors" is
\[
\langle f | g \rangle = \int f^*(x) g(x) \, dx.
\]

\( \frac{d^2}{dx^2} \) is an operator (analog of a matrix) transforming
one vector into another. \( \frac{d^2}{dx^2} \) can be shown to be
Hermitian, i.e. satisfying
\[
\int f^*(x) \frac{d^2}{dx^2} g(x) \, dx = \int g^*(x) \frac{d^2}{dx^2} f(x) \, dx.
\]
So according to the usual rules of matrices, the
eigenvectors of \( \frac{d^2}{dx^2} \) are complete and orthogonal.
These are \( \sin(n \pi x / L) \) and \( \cos(n \pi x / L) \).

The same is true of the Legendre polynomials,
which have a more complex-looking operator
\[
(1-x^2) \frac{d^2}{dx^2} - 2x \frac{d}{dx} + n(n+1)
\]
\( \star \) by integrating by parts and, to be really precise,
using some information about behavior of functions on
boundaries \( x=0, L \) \( \star \) periodicity.
\[ g(x=1, t) = \frac{1}{\sqrt{1-2t+t^2}} = \frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots \]

\[ = \sum_{n=0}^{\infty} p_n(1) t^n \]

Clearly, \( p_n(1) = 1 \).

Similarly, setting \( x = 0 \)

\[ g(x=0, t) = \frac{1}{\sqrt{1+t^2}} = (1+t^2)^{-1/2} \]

\[ = 1 - \frac{1}{2} t^2 + \frac{1}{8} \left(-\frac{1}{2}\right) \left(-\frac{3}{2}\right) t^4 + \frac{1}{16} \left(\frac{1}{3}\right) \left(-\frac{1}{2}\right) \left(-\frac{5}{2}\right) t^6 + \ldots \]

\[ = 1 - \frac{1}{2} t^2 + \frac{3}{8} t^4 - \frac{5}{16} t^6 + \ldots \]

We see that \( p_n(0) = 0 \) for all odd \( n \).

This is consistent with the fact that \( p_n(x) \) are odd functions of \( x \): \( p_n(x) = -p_n(-x) \) for \( n \) odd.

We also see \( p_0(0) = 1 \) \( p_2(0) = -\frac{1}{2} \)

\[ p_4(0) = \frac{3}{8} \quad p_6(0) = -\frac{5}{16} \]

and, again, we could write a general expression if we were really motivated to do so.
\( V(x,y) \) obeys the Laplace eqn

\[
\nabla^2 V = \left( \frac{d^2}{dx^2} + \frac{d^2}{dy^2} \right) V = 0
\]

Guess a solution \( V(x,y) = r(x) s(y) \) \( \Rightarrow \) "separation of variables"

\[
s(y) \frac{d^2 r}{dx^2} + r(x) \frac{d^2 s}{dy^2} = 0
\]

\[\Rightarrow\]
\[
\frac{1}{r(x)} \frac{d^2 r}{dx^2} = -\frac{1}{s(y)} \frac{d^2 s}{dy^2} = -k^2
\]

function of \( x \) only \( \Rightarrow \) function of \( y \) only \( \Rightarrow \) must be constant

\[
\text{Clearly, } \quad r(x) = \frac{\sin kx}{\cos kx}, \quad s(y) = \frac{\sinh ky}{\cosh ky}
\]

Since \( V(x,y) \) vanishes at \( x = 0 \) and \( x = b \)

we much choose \( \sin kx \) for \( r(x) \) and also \( k = \frac{n\pi}{b} \)

\[
V(x,y) = \sum_{n=1}^{\infty} q_n \frac{\sin \frac{n\pi x}{b}}{\sinh \frac{n\pi y}{b}}
\]

"superposition"

where I also used fact that \( V \) vanishes at \( y = 0 \)

to eliminate \( \cosh ky \) soln of \( s(y) \)
We use our final body information

\[ V(x, y=b) = f(x) = \sum \frac{q_n \sin \frac{n\pi x}{b} \sinh \frac{n\pi h}{b}}{n} \]

Multiply both sides by \( \sin \frac{n\pi x}{b} \) and integrate \( \int_0^b dx \)

Using orthogonality, \( \int_0^b \sin \frac{n\pi x}{b} \sin \frac{m\pi x}{b} dx = \frac{b}{2} \delta_{nm} \)

\[ \int_0^b \sin \frac{n\pi x}{b} f(x) dx = q_n \frac{b}{2} \sinh \frac{n\pi h}{b} \]

This determines the \( q_n \) which we put back in \( V(xy) \)

\[ V(xy) = \sum \frac{2}{b \sinh \frac{n\pi h}{b}} \int_0^b \sin \frac{n\pi x'}{b} f(x') dx' \]

Reorganizing

\[ V(xy) = \int_0^b f(x') g(x, x', y) dx' \]

with Green's function

\[ g(x, x', y) = \sum \frac{2}{b \sinh \frac{n\pi h}{b}} \sin \frac{m\pi x'}{b} \sin \frac{m\pi x}{b} \sinh \frac{m\pi h}{b} \]
We have \( V(x, y) \) obeying Laplace Eqn

\[
p^2 V = \left( \frac{d^2}{dx^2} + \frac{1}{y^2} \right) V(x, y) = 0
\]

Try separation of variables \( V(x, y) = r(x)s(y) \)

\[
s(y) \frac{d^2r}{dx^2} + \frac{r(x)}{y^2} \frac{d^2s}{dy^2} = 0
\]

\[
\frac{1}{r(x)} \frac{d^2r}{dx^2} = - \frac{1}{s(y)} \frac{d^2s}{dy^2} = - k^2
\]

Function of \( x \) only \( \Rightarrow \) function of \( y \) only \( \Rightarrow \) must be constant

We have \( r'(x) = e^{ikx} \)

\[
s(y) = e^{kY} \leftarrow \text{eliminate in} \quad e^{-kY} \quad \text{upper half plane}
\]

See p6-2 for important detail. \( \Rightarrow \) to avoid \( V \) diverging

\[
V(x, y) = \int_{-\infty}^{\infty} a(k) e^{ikx} e^{-kY} dk \quad \text{"superposition"}
\]

We are given

\[
V(x, y = 0) = f(x) = \int_{-\infty}^{\infty} a(k) e^{ikx} dk
\]

Invert this Fourier integral \( \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = a(k) \)
Putting together

\[ V(x, y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} f(x') e^{-ikx} dx' e^{-iky} \]

\[ = \int dx' f(x') g(x, x', y) \]

with \( g(x, x', y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')} e^{-ky} \)

In this case we can explicitly compute \( g(x, x', y) \)

Note that in selecting \( e^{-ky} \) over \( e^{ky} \) we were assuming \( k > 0 \). We really meant \( e^{-1k|y|} \)

\[ g(x, x', y) = \int_{-\infty}^{0} \frac{dk}{2\pi} e^{ik(x-x')} e^{ky} + \int_{0}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')} e^{-ky} \]

\[ = \frac{1}{2\pi} \left\{ \frac{e^{ky+ik(x-x')}}{y+i(x-x')} \bigg|_0^\infty + \frac{e^{-ky+ik(x-x')}}{-y+i(x-x')} \bigg|_0^{-\infty} \right\} \]

\[ = \frac{1}{2\pi} \left\{ \frac{1}{y+i(x-x')} - \frac{1}{-y+i(x-x')} \right\} \]

\[ = \frac{1}{2\pi} \left\{ \frac{y-i(x-x') + y+i(x-x')}{y^2 + (x-x')^2} \right\} \]

\[ g(x, x', y) = \frac{y/\pi}{y^2 + (x-x')^2} \]
If \( f(x) = V_0 \) is constant

\[
V(x, y) = \int_{-\infty}^{0} dx' V_0 \frac{y/\pi}{y^2 + (x-x')^2}
\]

change variables \( u = -x + x' \)
\[
du = dx'
\]

\[
V(x, y) = \int_{-\infty}^{0} V_0 \frac{y/\pi}{y^2 + u^2} \, du
\]

and trig substitution \( u = y \tan \theta \)
\[
du = y \sec^2 \theta \, d\theta
\]
\[
y^2 + u^2 = y^2 (1 + \tan^2 \theta) = y^2 \sec^2 \theta
\]

\[
V(x, y) = \int_{-\pi/2}^{\pi/2} V_0 \frac{y/\pi y \sec^2 \theta \, d\theta}{y^2 \sec^2 \theta}
\]

\[
= \frac{V_0}{\pi} \int_{-\pi/2}^{\pi/2} d\theta = V_0
\]

The potential is constant in the entire upper half plane.