The Schrödinger eqn

\[ \hat{H}\psi = \left[ \frac{\hat{p}^2}{2m} + V \right] \psi = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V \right] \psi = E\psi \]

is invariant under global charges of phase of \( \psi \)

\[ \psi \rightarrow \psi' \equiv e^{i\Lambda} \psi \quad \Lambda = \text{independent} \]

Notice \( p(\mathbf{x}) = |\psi(\mathbf{x})|^2 \) is also unchanged.

This invariance of \( p(\mathbf{x}) \) also holds true under local charges

\[ \psi \rightarrow \psi' \equiv e^{i\Lambda(\mathbf{x})} \psi \]

It is more convenient to write this as (more later on this...)

\[ \psi \rightarrow \psi' \equiv e^{-ie/\hbar \Lambda(\mathbf{x})} \psi \]

Now the Schrödinger eqn is invariant if we replace

\[ \hat{H} = \left[ \frac{(\hat{p} - eA)^2}{2m} + V \right] \]

and require \( \mathbf{A} \rightarrow \mathbf{A} + \nabla \Lambda \) when we change \( \psi \rightarrow e^{i\Lambda} \psi \).
The proof is simple

\[(p \cdot eA') \cdot \psi' = \left[ \frac{\hbar}{i} p - e(A + \nabla \lambda) \right] e^{\frac{i \epsilon_n}{\hbar} \psi} \]

\[= e^{\frac{i \epsilon_n}{\hbar} \frac{\hbar}{i} \nabla \psi} + e^{\frac{i \epsilon_n}{\hbar} \psi} - e^{\frac{i \epsilon_n}{\hbar} \psi} - e^{\frac{i \epsilon_n}{\hbar} \psi} \]

\[\text{cancel}\]

\[= e^{\frac{i \epsilon_n}{\hbar} (p - eA) \cdot \psi} \]

Thus appears also in \(V \psi\) and \(E \psi\) terms,

and hence cancels throughout.

The choice \(e^{i \epsilon_n \lambda / \hbar}\) is also motivated dimensionally

\([B] = \frac{\text{FORCE}}{\text{VELOCITY}} = \frac{ML}{AT} = \frac{M}{TE} \]

Since \(B = \lambda \times A\), \([A] = [B]L = ML/TQ\)

Since \(A + \lambda A + \nabla A\), \([\lambda] = [A]L = [B]L^2 = ML^2/TQ = [\phi_B]\)

Finally, \([e/\hbar] = \omega/ML^2/T = \Omega^2/ML^2 = 1/[\phi_B]\)

So \(e/\hbar\) is dimensionless.
SI Units for E, B

\[ [E] = \text{Force over } q = \frac{ML}{T^2 Q} \]

\[ E = -\frac{\dot{V}}{V} \]

\[ [V] = [E] \cdot L = \frac{ML^2}{T^2 Q} \]

check \[ [V] Q = [\text{Energy}] = \frac{ML^2}{T^2} \]

\[ [B] = \text{Force over } q \cdot L/T = \frac{ML}{T^2 Q} \cdot L = \frac{M}{TQ} \]

\[ \vec{B} = \vec{V} \times \vec{A} \]

\[ \vec{A} = [B] \cdot L = \frac{ML}{TQ} \]

\[ \vec{A} = \vec{A} + q \vec{V} \]

so a dimensionless \[ [\Phi] \] would be \[ \frac{TQ}{ML^2} [\Phi] \]

\[ [\Phi_B] = [B] \text{ (area)} = \frac{ML^2}{TQ} = [\Phi] \]

SI flux quantum \[ \frac{[h]}{[e]} = \frac{[ML^2 T^2]}{Q} \]

\[ \frac{h}{\delta e} \]
Put another way, when a QM particle traverses a region of non-zero \( \mathbf{A} \) it picks up a phase factor due to \( \mathbf{A} \):

\[
\Delta \phi = \frac{\mathbf{A} \cdot d\mathbf{r}}{\hbar}
\]

This phase factor depends on choice of gauge, however, going around a closed loop does not depend on choice of gauge.

\[
\oint \mathbf{A} \cdot d\mathbf{r} = \int (\mathbf{\nabla} \times \mathbf{A}) \cdot d\mathbf{a}
\]

\[
\frac{\hbar}{i} \partial_t \psi = -i \mathbf{E} \psi
\]

\[
\psi = e^{i \mathbf{k} \cdot \mathbf{r}} \quad \mathbf{E} = \frac{\mathbf{k} \times \mathbf{k}}{2m}
\]

as particle moves the phase changes from \( \mathbf{k} \cdot \mathbf{r}_1 \) to \( \mathbf{E} \cdot \mathbf{r}_2 \)
One of most interesting aspects of magnetism:

"magnetic order." Today's story is about role of
gauge degrees of freedom in magnetic order.

\[ \hat{\theta} \rightarrow \hat{\varphi} \]

\[ \text{magnetic moment} \]

in solids, e.g.

spins \( \leftrightarrow \) or

orbital moment \( \leftrightarrow \)

interaction, thermal

fluctuations,

quantum fluctuations

\[ [\hat{J}_1, \hat{J}_2] \rightarrow \text{just as you have uncertainty} \]

principle in particle positions

\[ [\hat{S}_x, \hat{S}_y] \rightarrow \text{so be in the orientation of} \]

their magnetic moment.

You studied \[ E = -J \sum \cos(\theta_i - \theta_j) \]

a classical model of magnetism (no quantum fluctuations)

\[ \theta_i = \theta_0 \quad \forall i \quad \text{minimized} \quad <E> = -NJ \]
The discussion of gauge degrees of freedom

EM might leave the impression that the presence of a freedom in the choice of potential does not affect the physics. You choose a gauge but whatever you choose to calculate in, the physics is the same. This is true

However, the extra degrees of freedom can have an effect at finite temperature, since they increase the possible configurations accessible to the system, hence the entropy, and decrease the likelihood of order.

Consider the Ising model \( S_i = \pm 1 \) on each site \( i \):

\[
E = -J \sum_{\langle ij \rangle} S_i S_j
\]

Two ground states

\[
\begin{array}{c|c|c|c}
\text{neighbors} & + & - & + \quad + J \\

total & + & + & + J
\end{array}
\]

Nature minimizes

\[
E_0 = -J N \frac{q}{2}
\]

Free energy

\[
P = E - TS
\]

\( q = \# \text{ neighbors} \)

(coordination \( q \))
Many more (exponentially more!) high energy states

\[ + - - + \]
\[ - + + - \]
\[ + - + - \]
\[ + - - + \]

\[ F = E - TS \]

Maximize \( F \)

Minimize \( E \)

\[ \beta \] is knob controlling relative weight of \( E \) vs. \( S \) in minimizing \( F \)

\[ T \]

Transition \( \Rightarrow \) sharp

Minimize \( E \)

\( T \) not \( T_c \)

Aside: Zee's Model. Transfer matrix in one dimension:

\[ Z = \sum \sum e^{-\beta E} = \sum \sum e^{\beta j S_1 S_2} e^{\beta j S_2 S_3} \]

\[ M = \begin{pmatrix} e^{\beta J} & e^{\beta J} \\ e^{-\beta J} & e^{-\beta J} \end{pmatrix} \]

\[ M_{S_1 S_2} M_{S_2 S_3} \]

\[ = \sum \sum M^2 (S_1, S_3) \]

\[ = \sum \sum M^{N (S_1, S_1)} \]

\[ \text{Eigenvalues of } M \text{ are } \lambda_{1,2} = e^{2 \cosh \beta J} ; \text{ } 2 \sinh \beta J \]
The Ising model has a global spin flip symmetry:

\[ S_i = -S_i \text{ on all } i \text{ simultaneously} \Rightarrow E \text{ is unchanged} \]

Evolves like

\[ \Psi(x) \rightarrow e^{-\frac{E}{k_B T}} \Psi(x) \]

at all \( x \) simultaneously

\[ E = -J \sum_{\langle i,j \rangle} (-S_i \cdot S_j) \]

Global discrete symmetry \( \mathbb{Z}_2 \)

XX model has global \( U(1) \) symmetry

Conventional Ising Model \( S_i = \pm 1 \) live on sites of lattice

\( E \) involves \( S_i \cdot S_j \) \( \langle i, j \rangle \) neighbors on bond

Wegner model \( J = \pm 1 \) live on bonds of lattice

This model has a local spin flip symmetry:

Pick a lattice site \( P \) and \( \tilde{S}_P \rightarrow -\tilde{S}_P \) for all spins on bonds emanating from \( P \).
The additional freedom for local gauge transformations inhibits phase transitions, conventional Ising law phase transition in d=2, Wegner does not (requires d=3).

What can you meaningfully measure in Wegner?

Product of $\mathbf{S}_q$ around closed loop (arbitrary size)
is "gauge invariant".

All this has close analogs in quantum field theory:

\[ \{ E, B \} \quad \text{mediated electric forces charged} \]
\[ \{ A, \phi \} \quad \text{AED particles (electrons)} \]
\[ \{ g_\mu, \gamma \} \quad \text{mediated strong forces strongly interact} \]
\[ \{ \text{fields} \} \quad \text{particles (quarks)} \]

Closed loops in Wegner $A-B$ confine in QCD