Completeness

Orthogonality of Bessel Functions

\[ \left[ \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + \left( k^2 x^2 - n^2 \right) \right] J_n(kx) = 0 \]

Recall Sturm-Liouville Theory

\[ p_0(x) \frac{d^2}{dx^2} + p_1(x) \frac{d}{dx} + p_2(x) \]

is Hermitian if \( p_1(x) = p_0'(x) \)

In such a situation, the eigenfunctions \( f(x) \)

\[ p f_\lambda(x) = \lambda f_\lambda(x) \]

are a complete set:

\[ g(x) = \sum \alpha_\lambda c(\lambda) f_\lambda(x) \quad \text{any } g \text{ can be expanded in } f_\lambda \]

and \( f_\lambda \) are also orthogonal:

\[ \int dx f_\lambda(x) f_{\lambda'}(x) = \delta_{\lambda,\lambda'} \text{ or } \delta(\lambda - \lambda') \]

\( \delta \) discrete or \( \delta(\lambda - \lambda') \) continuous.
The Bessel differential operator is Hermitian only after division by the "weight function" \( w(x) = x \)

\[
x \frac{d}{dx} x \frac{d}{dx} + \frac{1}{x} + (k^2 - \frac{n^2}{x}) \quad J_n(kx) = 0
\]

\( p_0(x) = x \quad p_1(x) = 1 \quad p_1' = p_0' \)

In this case, as we discussed, the eigenfunctions obey a generalized orthogonality

\[
\int w(x) f_\lambda(x) f_\mu(x) dx = \delta_\lambda^\mu \quad \delta(x-\sigma)
\]

Let's work out how this occurs explicitly for Bessel functions. Consider problem where we require function \((x \to p)\) to vanish at \( p = a \). When we solved Schrödinger, we saw this quantized the \( k \) values

\[
J_n(\alpha_n a) \quad \text{where } \alpha_n \text{ are roots of } J_n(x) \quad \text{in } J_n(\alpha_n a) = 0
\]

Aside: Orthogonality depends not only on \( f \) but also on boundary conditions \( \frac{d^2 f}{dx^2} = -k^2 f \quad 0 < x < a \to \sin \frac{n \pi x}{a} \)

\[
f(x) = 0 \quad x = 0, a
\]
The derivation below follows very closely last quarter's derivation of the \( p_1 = p_0 \) Hermiticity condition.

\[
\frac{\partial}{\partial p} J_n (\alpha_{nm} \frac{P}{a}) + \frac{\partial^2}{\partial p^2} J_n (\alpha_{nm} \frac{P}{a}) + \frac{\alpha_{nm}^2 P}{a^2} - \frac{v^2}{\rho} \right] J_n (\alpha_{nm} \frac{P}{a}) = 0
\]

\[
\left[ \frac{\partial^2}{\partial p^2} J_n (\alpha_{nm} \frac{P}{a}) + \frac{\partial}{\partial p} J_n (\alpha_{nm} \frac{P}{a}) + \frac{\alpha_{nm}^2 P}{a^2} - \frac{v^2}{\rho} \right] J_n (\alpha_{nm} \frac{P}{a}) = 0
\]

.. \( J_n (\alpha_{nm} \frac{P}{a}) \)

and subtract

\[
J_n (\alpha_{nm} \frac{P}{a}) \frac{\partial}{\partial p} \left[ \frac{\alpha_{nm}^2 P}{a^2} \right] J_n (\alpha_{nm} \frac{P}{a})
\]

\[
- J_n (\alpha_{nm} \frac{P}{a}) \frac{\partial}{\partial p} J_n (\alpha_{nm} \frac{P}{a})
\]

Integrate from 0 to \( a \) and then integrate by parts.

The "integral term" vanishes because one has two identical pieces differing by a \( \pi \) sign. The "surface term" vanishes at \( p = 0 \) because of the \( P \) factor and at \( p = a \) because of \( \alpha_{nm} \) argument.

Thus as long as \( \alpha_{nm}^2 \neq \alpha_{nm}^2 \), we have

\[
\int_0^a \rho J_n (\alpha_{nm} \frac{P}{a}) J_n (\alpha_{nm} \frac{P}{a}) \, dp = 0
\]

**ORTHOGONALITY**

Normalisation

\[
\int_0^a \left[ J_n (\alpha_{nm} \frac{P}{a}) \right]^2 \, dp = \frac{a^2}{2} \left[ J_{n+1} (\alpha_{nm}) \right]^2
\]

**Exercise:** from recurrence reln (HW-1)
Summary: we can expand $f(p)$ over $p < a$ as $f(a) = 0$

$$f(p) = \sum_{n=1}^{\infty} C_{nm} J_n(\alpha_{nm} p/a)$$

$$C_{nm} = \frac{2}{a^2 (J_{n+1}(\alpha_{nm}))^2} \int_0^a f(p) J_n(\alpha_{nm} p/a) \, dp$$

Q: Why are $\{ J_n(\alpha_{nm} p/a) \}$ complete for each $n$?

Should we have to include all $n$ in expansion?

A: Different $\beta$, PDE for each $n$.