Assignment Two, Due Friday, February 3, 5:00 pm.

[0.] Problem Six from Assignment One.

[1.] Compute the potential $V(r, \theta, \phi)$ due to a thin disk of charge $Q$ of constant charge per area. Make a convenient choice of origin and orientation of your axes.

[2.] A point charge $Q$ is a distance $z_0$ from an infinite metallic conducting plane held at potential $V = 0$. Compute the induced charge density on the surface of the plane and the total charge on the plane. What is the electric field (magnitude and direction) at the surface of the plane?

[3.] A point charge $Q$ is a distance $z_0$ from the center of a metallic sphere of radius $R < z_0$ held at a potential $V = 0$. What is the Greens function for positions outside the sphere? What is the potential outside the sphere?
We first compute the potential along a high symmetry direction — the z axis. We do this by integrating the potential due to a collection of rings of charge which together build up the disk.

\[ V_{\text{ring}} = \frac{2\pi \rho \sigma}{4\pi \epsilon_0 \sqrt{\rho^2 + z^2}} \]

where

\[ \rho = \text{disk radius} \]
\[ \sigma = \text{charge on ring of radius } \rho \]
\[ \epsilon_0 = \text{distance of charge to point on } z \text{ axis} \]

\[ V_{\text{disk}} = \int_0^a \frac{V_{\text{ring}}}{d\rho} \]

\[ = \frac{\sigma}{2 \pi \epsilon_0} \int_0^a \frac{9 \rho}{\sqrt{\rho^2 + z^2}} \]

\[ = \frac{\sigma}{2 \pi \epsilon_0} \left[ \frac{(\rho^2 + z^2)^{1/2}}{1} \right]^a_0 \]

\[ = \frac{\sigma}{2 \pi \epsilon_0} \left\{ (a^2 + z^2)^{1/2} - z \right\} \]

Check this by considering \( \pm \frac{a}{2} \)

\[ (a^2 + z^2)^{1/2} = z \left( 1 + \frac{a^2}{2z^2} \right)^{1/2} \approx z \left( 1 + \frac{a^2}{2z^2} \right) \]
Thus for \( z > a \)

\[
V_{\text{disk}} \rightarrow \frac{\sigma}{2 \varepsilon_0} \left\{ z^2 + \frac{a^2}{2} - \frac{z}{2} \right\} = \frac{\sigma a^2}{4 z \varepsilon_0}
\]

Using \( Q = \pi a^2 \), \( V_{\text{disk}} = \frac{Q}{4 \pi \varepsilon_0 z} \)

which makes sense: at large distances disk appears as a point charge.

Meanwhile for \( z < a \)

\[
V_{\text{disk}} \rightarrow \frac{\sigma}{2 \varepsilon_0} \left\{ a + \frac{z^2}{2} - \frac{z}{2} \right\} \rightarrow V_0 - \frac{\sigma z}{2 \varepsilon_0}
\]

This also makes sense! The field \( E \) near the disk is that of an infinite plane \( E = \frac{\sigma}{2 \varepsilon_0} \). Since \( E_z = -\frac{\partial V}{\partial z} \), we must have \( V = -\frac{\sigma z}{2 \varepsilon_0} \).

The second part of the problem leverages the high symmetry solution to the entire upper half plane, i.e., off the \( z \)-axis. By symmetry (axi, radial)

\( V \) will depend only on \( r \) and \( \theta \) and we know

\[
V(r, \theta) = \sum_{n=0}^{\infty} [a_n r^n + b_n r^{-(n+1)}] P_n(\cos \theta)
\]

\( \theta = 0 \) and \( P_n(1) = 1 \)
\[
\sum_{l=0}^{\infty} \left[ q^l r^l + b^l r^{l+1} \right] \xi_l = \frac{\sigma}{2\varepsilon_0} \left\{ \left( a^2 + r^2 \right)^{1/2} - r \right\}
\]

Assuming we are interested in \( r > a \), right-hand side is

\[
\frac{\sigma}{2\varepsilon_0} \left\{ \frac{r}{(1 + a^2/2r^2)^{1/2}} - \frac{1}{r} \right\}
\]

\[
= \frac{\sigma}{2\varepsilon_0} \left\{ r \left[ 1 + \frac{a^2}{2r^2} - \frac{1}{8r^4} + \frac{1}{16} \frac{a^4}{r^6} \ldots \right] - \frac{1}{r} \right\}
\]

\[
= \frac{\sigma}{2\varepsilon_0} \left\{ \frac{a^2}{2r} - \frac{1}{16} \frac{a^4}{r^3} + \frac{a^6}{16r^5} \ldots \right\}
\]

We conclude that \( q_l = 0 \) \( \forall l \) and \( b_l = 0 \) for \( l \) odd.

And specifically,

\[
b_0 = \frac{\sigma a^2}{4\varepsilon_0} = \frac{\sigma q}{4\pi \varepsilon_0}
\]

\[
b_2 = -\frac{\sigma a^4}{16\varepsilon_0} = -\frac{\sigma q a^2}{16\pi \varepsilon_0}
\]

\[
b_4 = +\frac{\sigma a^6}{32\varepsilon_0} = +\frac{\sigma q a^4}{32\pi \varepsilon_0}
\]
Can satisfy body condition that \( V = 0 \) on conducting plane by placing \(-Q\) image charge on opposite side

\[
\begin{align*}
\vec{E}_+ &= \frac{Q}{4\pi\varepsilon_0} \frac{1}{x^2 + z_0^2} \hat{x} + \frac{y}{(x^2 + z_0^2)^{3/2}} \hat{y} \\
\vec{E}_- &= \frac{Q}{4\pi\varepsilon_0} \frac{1}{x^2 + z_0^2} \hat{x} - \frac{y}{(x^2 + z_0^2)^{3/2}} \hat{y}
\end{align*}
\]

\[
\vec{E}_{\text{tot}} = \frac{-Q z_0}{2\pi\varepsilon_0 (x^2 + z_0^2)^{3/2}}
\]

In general \( \phi \) if not on \( x \) ax.

Charge density on plane related to \( \vec{E} \) by Gauss' Law.

\[
\phi_F = |E| A = \frac{Q}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}
\]

\[
\Rightarrow \quad \sigma = \varepsilon_0 |E|
\]

Here \( \sigma \) is density. \( E_z < 0 \)
Integrate $t$ to get total change $Q_p$ on plane.

\[ Q_p = \int_0^\infty 2\pi p dp \, V = \int_0^\infty 2\pi p dp \left[ -\frac{\alpha z_0}{2\pi z_0 \left( p^2 + z_0^2 \right)^{3/2}} \right] \]

\[ \downarrow \]

\[ = -\alpha z_0 \int_0^\infty p dp \left\{ \frac{1}{(p^2 + z_0^2)^{1/2}} \right\} \]

\[ = -\alpha z_0 \left\{ 0 + \frac{1}{2} z_0 \right\} = -\alpha \]

\[ Q_p = -\alpha \quad \text{exactly matches force change.} \]
It is probably more consistent with our usual notation to denote the position of the point charge as \( r' \) rather than \( r \).

We want to find the location \( r_\pm \) and magnitude \( Q' \) which will give \( V = 0 \) on the surface of the sphere.

Clearly \( Q' < 0 \) and also its location, by symmetry, must be on the line connecting sphere center and charge +Q.

\[
0 = V = \frac{Q}{4\pi\epsilon_0} \left( \frac{1}{r_+} - \frac{1}{r_-} \right) \quad \alpha = \frac{Q'}{Q}
\]

\[
r_+ = \sqrt{R^2 + r'^2 - 2Rr'\cos\theta}
\]

\[
r_- = \sqrt{R^2 + r'^2 - 2Rr'\cos\theta}
\]
To get this satisfied \( \frac{1}{r^2} = \frac{\alpha}{r^2} \)

\[
R^2 + r_i^2 = \alpha \left( r^2 + r_i^2 \right)
\]

This is true if

\[
2Rr_i = \alpha^2 2Rr_i
\]

The bottom eqn gives \( r_i = \alpha^2 r_i \) and hence

\[
R^2 + r_i^2 = \frac{r_i}{r'} (R^2 + r_i^2)
\]

\[
r_i^2 = \frac{R^2 + r_i^2}{r'} r_i + R^2 = 0
\]

\[
r_i = \frac{1}{2} \left\{ \frac{R^2 + r_i^2}{r'} \pm \sqrt{\left( \frac{R^2 + r_i^2}{r'} \right)^2 - 4R^2} \right\}
\]

\[
= \frac{1}{r_i^{1/2}} \left\{ R^4 + 2R^2 r_i^2 + r_i^4 - 4R^2 r_i^2 \right\}^{1/2}
\]

\[
= \frac{1}{r_i^{1/2}} \left( R^2 - r_i^2 \right)^2
\]

\[
r_i = \frac{1}{2} \frac{1}{r'} \left\{ \left( R^2 + r_i^2 \right) \pm \sqrt{R^2 - r_i^2} \right\}
\]
The solution gives \( r_\pm = r' \) and then \( \alpha = \lambda \). There is an (expected) trivial solution. One can get \( V = 0 \) on the sphere surface by placing \(-\lambda (\alpha = 1)\) right on top (\( r_\pm = r' \)) of \(+\lambda\). In fact \( V = 0 \) everywhere.

The \(+\lambda\) solution is non-trivial, giving
\[
\frac{r_\pm}{R/r'} \quad \rightarrow \quad r_\pm < R \text{ since } R/r' < 1
\]
\[
\alpha = \frac{R}{r'} \quad \rightarrow \quad \alpha < 1 \text{ since } R/r' < 1
\]

Then the Greens function is
\[
G(r, r') = \frac{1}{|r-r'|} - \frac{R/r'}{|r-(R/r')r'|}
\]

To get the potential repeat the notation in problem statement

and place \(+\lambda\) at \( \vec{\omega} \)
\[
V(\vec{r}) = \frac{\lambda}{4\pi \varepsilon_0} \left( \frac{1}{|\vec{r}-\vec{\omega}|} - \frac{R/\lambda}{|\vec{r}-R^2/\lambda^2 \cdot \vec{\omega}|} \right)^2
\]