Summary of Procedure to Add Angular Momentum + Further General Discussion

(1) We never ask how to add linear momentum because there are no nonzero commutation relations to make anything but the "obvious" result true: If particle 1 has linear momentum $\vec{p}_1 = (p_{1x}\ p_{1y}\ p_{1z})$ and particle 2 has $\vec{p}_2 = (p_{2x}\ p_{2y}\ p_{2z})$ then $\vec{P}_{tot} = \vec{p}_1 + \vec{p}_2 = (p_{1x}+p_{2x}, p_{1y}+p_{2y}, p_{1z}+p_{2z})$

Angular momentum is different because $[L_x, L_y] = i\hbar L_z$ etc!

(2) Given particle 1 with $L_1^2 = l_1(l_1+1)\hbar^2$ and $L_{z1} = m_1\hbar$ $(m_1 = -l_1, -l_1+1, \ldots +l_1)$ and particle 2 with $L_2^2 = l_2(l_2+1)\hbar^2$ and $L_{z2} = m_2\hbar$ $(m_2 = -l_2, -l_2+1, \ldots +l_2)$ then $(L_1+L_2)^2 = l(l+1)\hbar^2$ with $l = l_1+l_2, l_1+l_2-1, \ldots |l_1-l_2|$. For each l, $m = -l, -l+1, \ldots +l$

(3) We saw two examples of this in class, and saw the "counting" worked out, ie the dimensions of the Hilbert spaces using $L_1^2, L_2^2, L_{1z}, L_{2z}$ as complete set of commuting operators and $L_1^2, L_2^2, (L_1+L_2)^2, L_{1z}+L_{2z}$ agreed.

$\ell_1 = \ell_2 = 1/2 \qquad (2\ell_1+1) = 2 \qquad (2\ell_2+1) = 2$

$2 \times 2 = 4$ dim Hilbert space

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$\left.\begin{array}{ll} \ell = 1 & (2\ell+1) = 3 \\ 0 & (2\ell+1) = 1 \end{array}\right\}$ 4 dim Hilbert space

$\ell_1 = \ell_2 = 1 \qquad (2\ell_1+1) = 3 \qquad (2\ell_2+1) = 3$

$3 \times 3 = 9$ dim Hilbert space

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$\left.\begin{array}{ll} \ell = 2 & 2\ell+1 = 5 \\ 1 & 2\ell+1 = 3 \\ 0 & 2\ell+1 = 1 \end{array}\right\}$ 9 dim Hilbert space

You did $1 + 1/2$ in Homework.

To add angular momentum start with the state of largest l and largest m. It is formed by using the states of largest m_1 and m_2. That is

$$|l = l_1 + l_2 \;\; m = l_1 + l_2\rangle = |l_1 \;\; m_1 = l_1\rangle \, |l_2 \;\; m_2 = l_2\rangle$$

largest m possible

largest m_1

largest m_2

Keep apply $L_- = L_{1-} + L_{2-}$ to this state

Since $[L_-, L^2] = 0$ you will not change $l = l_1 + l_2$ when you do this. However you will systematically decrease m from $l_1 + l_2$ to $l_1 + l_2 - 1$ to $l_1 + l_2 - 2$ all the way to $-l_1 - l_2$.

Your next step is to form the state of largest m which has $l = l_1 + l_2 - 1$. You do this by combining $|l_1 \;\; m_1 = l_1\rangle \, |l_2 \;\; m_2 = l_2 - 1\rangle$ and $|l_1 \;\; m_1 = l_1 - 1\rangle \, |l_2 \;\; m_2 = l_2\rangle$ in a way which is orthogonal to $|l = l_1 + l_2 \;\; m = l_1 + l_2 - 1\rangle$

From this state you can build all $l = l_1 + l_2 - 1$ states by apply L_-.

This process is continued until you exhaust all possibilities down to $|l_1 - l_2|$

Work again through examples $l_1 = l_2 = 1$ $l_1 = l_2 = 1/2$ $l_1 = 1$ $l_2 = 1/2$ to see how it works.

A "picture" is on the next page

(I have not normalized the states)

$$|l = l_1 + l_2 \;\; m = l_1 + l_2\rangle = |l_1\, l_1\rangle\, |l_2\, l_2\rangle$$

$$\Big\downarrow L_-$$

$$|l = l_1 + l_2 \;\; m = l_1 + l_2 - 1\rangle = \sqrt{2l_1}\, |l_1\, l_1 - 1\rangle\, |l_2\, l_2\rangle + \sqrt{2l_2}\, |l_1\, l_1\rangle\, |l_2\, l_2 - 1\rangle \xrightarrow{\text{orthogonality}}$$

$$|l = l_1 + l_2 - 1 \;\; m = l_1 + l_2 - 1\rangle = \sqrt{2l_2}\, |l_1\, l_1 - 1\rangle\, |l_2\, l_2\rangle - \sqrt{2l_1}\, |l_1\, l_1\rangle\, |l_2\, l_2 - 1\rangle$$

$$\Big\downarrow L_- \qquad\qquad \Big\downarrow L_-$$

$$|l = l_1 + l_2 \;\; m = l_1 + l_2 - 2\rangle = \ldots \qquad |l = l_1 + l_2 - 1 \;\; m = l_1 + l_2 - 2\rangle = \ldots \xrightarrow{\text{orthogonality}} |l = l_1 + l_2 - 2 \;\; m = l_1 + l_2 - 2\rangle = \ldots$$

$$\Big\downarrow L_- \qquad\qquad \Big\downarrow L_- \qquad\qquad \Big\downarrow L_-$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

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