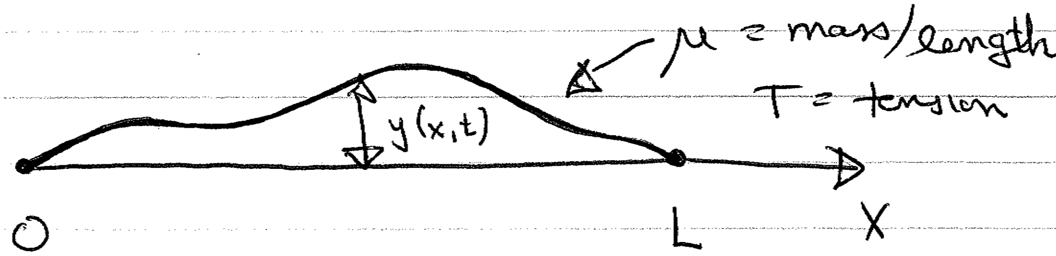
VW-1

Vibrating Wave

Perturbation Theory ideas can also be applied to classical mechanics, eg suppose we want to compute frequency shift due to attaching a small weight to a piano string



Wave Eqn $\quad v^2 \dfrac{\partial^2 y}{\partial x^2} = \dfrac{\partial^2 y}{\partial t^2} \qquad v = \sqrt{\dfrac{T}{\mu}}$

✓ units $\quad T = \dfrac{ML}{t^2} \qquad \mu = \dfrac{M}{L}$

separation of variables

$$\Longrightarrow \quad \sin\frac{n\pi x}{L} \quad \left\{ \begin{matrix} \cos\dfrac{n\pi v t}{L} \; a_n \\ \sin\dfrac{n\pi v t}{L} \; b_n \end{matrix} \right\}$$

normalization $\sqrt{\dfrac{2}{L}}$ ← why normalize in VW case?!

VW-2

$$y(x,t) = \sum_n \sin\frac{n\pi x}{L}\left\{a_n \cos\frac{n\pi v t}{L} + b_n \sin\frac{n\pi v t}{L}\right\}$$

a_n, b_n determined by

initial position $y(x,0)$ and velocity $\frac{\partial y}{\partial t}(x,0)$

Frequency $\omega_n = v k = v\frac{n\pi}{L}$

$y = f(x)\,g(t)$

$v^2 f'' g = f g'' =$

$$v^2 \frac{f''}{f} = \frac{g''}{g} = -\omega^2 \qquad g = \sin \omega t$$

$$\cos \omega t$$

$$f = \sin\left(\frac{\omega}{v}\right)x \quad k$$

$$\cos\frac{\omega}{v}x \quad k$$

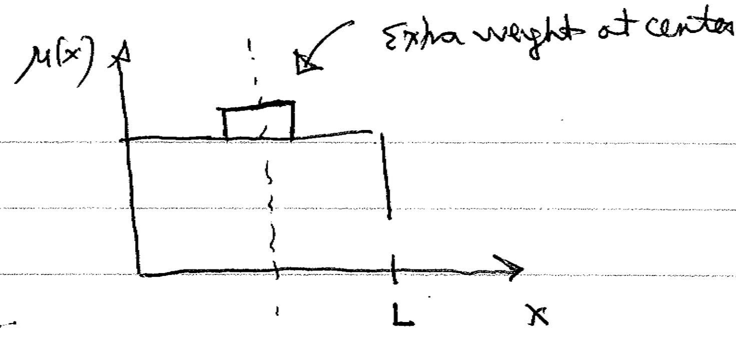
$v^2 f'' = -\omega^2 f$ like time indep Sch Eqn

$$\frac{T}{\mu} f'' = -\omega^2 f$$

shift in $-\omega^2$

due to μ

$\boxed{\text{VW - 3}}$



$$\mu_0 \to \mu_0 + \lambda \qquad |x - L/2| \le \delta\ell$$
$$\mu_0 \qquad |x - L/2| > \delta\ell$$

Might expect to affect only $n = 1, 3, 5, \ldots$
modes which do not have nodes at $L/2$ (in limit $\delta\ell \to 0$)

$$\frac{T}{\mu_0} \to \frac{T}{\mu_0 + \lambda} = \frac{T}{\mu_0}\left(1 + \frac{\lambda}{\mu_0}\right)^{-1} \approx \frac{T}{\mu_0}\left(1 - \frac{\lambda}{\mu_0}\right) = \frac{T}{\mu_0} - \frac{T}{\mu_0}\frac{\lambda}{\mu_0}$$

"perturbation" is $-\frac{d^2}{dx^2}\frac{T}{\mu_0}\frac{\lambda}{\mu_0}$

$$\delta(-\omega_n^2) = \langle\psi_n|V|\psi_n\rangle = \int_{-L/2-\delta\ell}^{L/2+\delta\ell} dx \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\ \frac{-T\lambda}{\mu_0^2}\frac{d^2}{dx^2}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{T\lambda 2}{\mu_0^2 L}\int_{-L/2-\delta\ell}^{L/2+\delta\ell}\left(-\frac{n^2\pi^2}{L^2}\right)\sin^2\frac{n\pi x}{L}\,dx$$

$$\delta(\omega_n^2) = -\frac{2\lambda T}{\mu_0^2 L^3}n^2\pi^2\int_{+L/2-\delta\ell}^{L/2+\delta\ell}\frac{1}{2}\left(1 - \cos\frac{2n\pi x}{L}\right)dx$$

expected ω goes down (for $\lambda > 0$)

Recall $\omega_{n,0}^2 = \frac{T}{\mu_0}\frac{n^2\pi^2}{L^2}$

$$\delta(\omega_n^2) = -\frac{\lambda}{\mu_0 L}\omega_{n,0}^2\left[x - \frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\right]_{L/2-\delta\ell}^{L/2+\delta\ell}$$

$$= -\frac{\lambda}{\mu_0}\omega_{n,0}^2\frac{1}{L}\left\{2\delta\ell - \frac{L}{2n\pi}\left(\sin\left(n\pi + \frac{2n\pi\delta\ell}{L}\right) - \sin\left(n\pi - \frac{2n\pi\delta\ell}{L}\right)\right)\right\}$$

$$\left((-1)^n\frac{2n\pi\delta\ell}{L} + (-1)^n\frac{2n\pi\delta\ell}{L}\right)$$

using $\sin\theta \sim \theta$

$$= -\frac{\lambda}{\mu_0}\omega_{n,0}^2\frac{1}{L}\left[2\delta\ell - 2(-1)^n\delta\ell\right]$$

$$= -\frac{\lambda\delta\ell}{\mu_0 L}\omega_{n,0}^2\begin{cases}4 & n \text{ odd}\\ 0 & n \text{ even}\end{cases}$$

$$\delta(\omega_n^2) = -\frac{2\delta m}{M}\omega_{n,0}^2$$

for n odd only

Even more crude approximation

$$\mu_0 \to \mu_0 + \frac{\delta m}{L} \quad \underline{\underline{\text{uniform}}}$$

$$\cancel{\delta(\omega_n^2)} \quad \omega_n^2 = \frac{T}{\mu}\frac{n^2\pi^2}{L^2}$$

$$= \frac{T}{\mu_0}\left(1 - \frac{\delta m}{L\mu_0}\right)\frac{n^2\pi^2}{L^2}$$

$$\omega_n^2 = \omega_{n,0}^2 - \underbrace{\frac{T}{\mu_0}\frac{n^2\pi^2}{L^2}}_{\omega_{n,0}^2}\ \underbrace{\frac{\delta m}{\mu_0 L}}_{\delta m/M}$$

$$\delta(\omega_n^2) = -\frac{\delta m}{M}\omega_{n,0}^2$$

misses the physics of node at $L/2$ for even modes!

Where does factor of 2 in "correct" perturbation theory come from?