4.

Perturbation theory at last

$$H = H^0 + \lambda H'$$

$$|\psi_n\rangle = |\psi_n^0\rangle + \lambda|\psi_n^1\rangle + \lambda^2|\psi_n^2\rangle + \dots$$

$$E_n = E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2 + \dots$$

$$H|\psi_n\rangle = E_n \psi_n$$

$$* \quad H^0|\psi_n^1\rangle + H'|\psi_n^0\rangle = E_n^0|\psi_n^1\rangle + E_n^1|\psi_n^0\rangle$$

$$** \quad H_0|\psi_n^2\rangle + H'|\psi_n^1\rangle = E_n^0|\psi_n^2\rangle + E_n^1|\psi_n^1\rangle + E_n^2|\psi_n^0\rangle$$

Take $\langle\psi_n^0|$ on *

$$\underbrace{\langle\psi_n^0|H^0|\psi_n^1\rangle}_{} + \langle\psi_n^0|H'|\psi_n^0\rangle$$

$$= E_n^0\langle\psi_n^0|\psi_n^1\rangle + E_n^1\underbrace{\langle\psi_n^0|\psi_n^0\rangle}_{1}$$

Hermitian

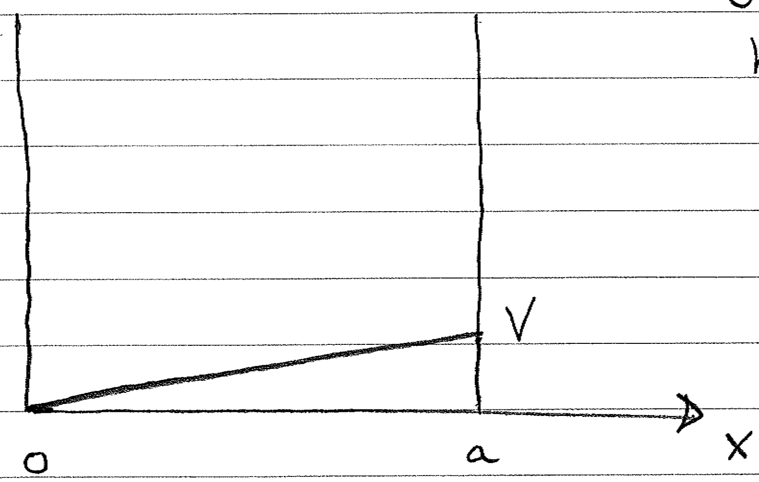
$$E_n^0\langle\psi_n^0|\psi_n^1\rangle$$ so first terms on lhs + rhs cancel

$$\langle\psi_n^0|H'|\psi_n^0\rangle = E_n^1 = \int dx\, \psi_n^{0*}(x) H' \psi_n^0(x)$$

"Physically Reasonable" Shift in Energy is expectation value of perturbation!

Read Example 6.1 ! We do something a bit harder

Square well with tilted base



Example 6.1

$H' = V$

$$E_n^1 = \frac{2}{a} V \underbrace{\int_0^a \sin^2 \frac{n\pi x}{a}\, dx}_{\frac{1}{2}a}$$

$= V$ ✓

$$\psi_n^0(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$$

$$E_n^1 = \int_0^a \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \; V\frac{x}{a} \; \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}\, dx$$

$$= \frac{2}{a}\frac{V}{a} \int_0^a x \sin^2 \frac{n\pi x}{a}\, dx \qquad y = \frac{n\pi x}{a}$$

$$x = \frac{ay}{n\pi}$$

$$= \frac{2V}{a^2} \int_0^{n\pi} y \sin^2 y\, dy \left(\frac{a}{n\pi}\right)^2$$

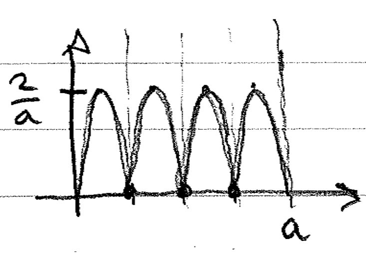
$$= \frac{2V}{n^2\pi^2} \int_0^{n\pi} y \sin^2 y\, dy$$

See Boring Integral page

$$= V/2$$

Reasonable ? average amount potential is raised

For $n=4$



$$|\psi_n^0(x)|^2 = \frac{2}{a} \sin^2 \frac{n\pi x}{a} = \text{probability}$$

BI - 1

Boring integral

$\cos^2 y - \sin^2 y = \cos 2y$

$\cos^2 y + \sin^2 y = 1$

$$\int_0^{n\pi} y \sin^2 y \, dy = \int_0^{n\pi} y \, \tfrac{1}{2}(1-\cos 2y)\, dy$$

$$= \frac{y^2}{4}\Big|_0^{n\pi} - \underbrace{\int_0^{n\pi} \overset{u}{y} \, \overset{dv}{\cos 2y \, dy}}$$

$$\underbrace{y}_{u}\,\underbrace{\frac{\sin 2y}{2}}_{v}\Big|_0^{n\pi} - \int_0^{n\pi} \underbrace{\frac{\sin 2y}{2} \, dy}_{v\ du}$$

both vanish

$$= \frac{n^2\pi^2}{4}$$

ASHO-1

Anharmonic oscillator

$$V(x) = \frac{1}{2} m \omega^2 x^2 + \frac{1}{4} A x^4$$

Is this a problem you can do classically?

$$m\ddot{x} = -m\omega^2 x - A x^3$$

$$x(t) = ?$$

$$E_n^1 = \langle \psi_n^0 | \frac{1}{4} A x^4 | \psi_n^0 \rangle$$

$$= \frac{1}{4} A \left(\frac{\hbar}{2m\omega}\right)^2 \langle \psi_n^0 | (a + a^+)^4 | \psi_n^0 \rangle$$

16 terms aaaa etc

Q: Do all survive? Only 6

$a a a^+ a^+$	$\sqrt{n+1}\,\lvert n+1\rangle$	$\sqrt{n+1}\sqrt{n+2}\,\lvert n+2\rangle$	$(n+1)(n+2)$
$a a^+ a a^+$	$\sqrt{n+1}\,\lvert n+1\rangle$	$\sqrt{n+1}\sqrt{n+1}\,\lvert n\rangle$	$(n+1)(n+1)$
$a^+ a a a^+$	$\sqrt{n+1}\,\lvert n+1\rangle$	$\sqrt{n+1}\sqrt{n+1}\; n$	$(n+1)\, n$
$a a^+ a^+ a$			$n(n+1)$
$a^+ a a^+ a$			$n\, n$
$a^+ a^+ a a$			$n(n-1)$

$$\frac{3}{4} A \left(\frac{\hbar}{2m\omega}\right)^2 (2n^2 + 2n + 1)$$

$$\left.\begin{array}{ll} n^2 + 3n + 2 & \\ n^2 + 2n + 1 & \\ n^2 + n & \\ n^2 + n & n^2 - n \\ n^2 & \end{array}\right\} \quad 6n^2 + 6n + 3$$

D1

δ-function potential $\quad -\alpha\,\delta(x)$

Exactly one bound state independent of α

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar}\, e^{-m\alpha|x|/\hbar^2}$$

$$E = -m\alpha^2/2\hbar^2$$

Idea is $\quad \psi(x) = A e^{-kx} \quad x>0$

$\qquad\qquad\qquad\;\; B e^{kx} \quad x<0$

so $A = B$

and then require ψ be continuous and

$$-\frac{\hbar^2}{2m}\int_{-\epsilon}^{\epsilon}\frac{d^2\psi}{dx^2}\,dx + \int_{-\epsilon}^{\epsilon} V(x)\psi(x)\,dx = E\underbrace{\int_{-\epsilon}^{\epsilon}\psi(x)\,dx}_{0}$$

$$-\frac{\hbar^2}{2m}\left[\frac{d\psi}{dx}\Big|_{\epsilon} - \frac{d\psi}{dx}\Big|_{-\epsilon}\right] - \alpha\psi(0) = 0$$

$$-\frac{\hbar^2}{2m}\left(-kA - kA\right] = \alpha A$$

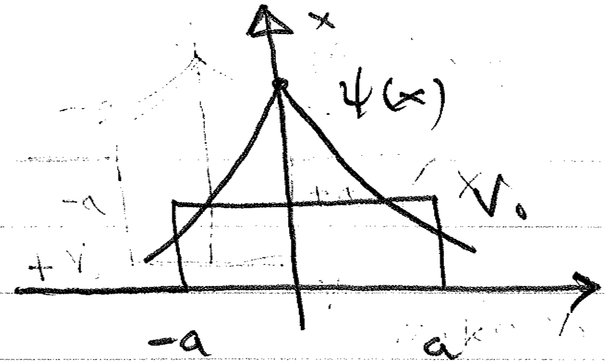
$$\Rightarrow \quad k = m\alpha/\hbar^2$$

Looking at Sch Eqn for $x \neq 0$ $\quad -\frac{\hbar^2 k^2}{2m} = E = -\frac{\hbar^2}{2m}\frac{m^2\alpha^2}{\hbar^4}$

$$E = -\frac{m\alpha^2}{2\hbar^2}$$

D2

$H' = +V_0 \quad |x|<a$



$$E_0' = \int_{-a}^{a} \frac{m\alpha}{\hbar^2} e^{-2m\alpha|x|/\hbar^2} (+V_0)\, dx$$

$$= 2V_0 \frac{m\alpha}{\hbar^2} \int_0^a e^{-2m\alpha x/\hbar^2} dx$$

$$= 2V_0 \frac{m\alpha}{\hbar^2}\left(\frac{-\hbar^2}{2m\alpha}\right) e^{-2m\alpha x/\hbar^2}\Big|_0^a$$

$$= V_0\left(1 - e^{-2m\alpha a/\hbar^2}\right)$$

"a" large $\Rightarrow$ entire $\psi_0(x)$ is inside $+V_0$ $\quad E_0' = V_0$

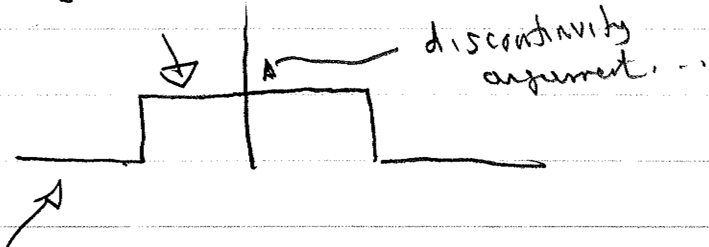
"a" small $\Rightarrow$ only part of $\psi_0(x)$ inside $+V_0$

$$E_0' = V_0\left[1 - (1 - 2m\alpha a/\hbar^2)\right]$$

$$= 2V_0 m\alpha a/\hbar^2$$

Is exact soln possible?

$e^{k'x}$ with $\frac{\hbar^2 k'^2}{2m} + V_0 = E$



e^{kx} with $\frac{\hbar^2 k^2}{2m} = E$