AMSB - 1

Stepping Back...

Is logic of adding angular momentum clear?

Consider solving $H|\psi\rangle = E|\psi\rangle$.

We choose a basis eg eigenstates $|x\rangle$ of position operator $\hat{X}$

$\hat{H}$ is not diagonal in that basis so we figure out

the linear combination of $|x\rangle$ which are

$$|\psi\rangle = \int dx \underbrace{\langle x|\psi\rangle}_{\psi(x)} |x\rangle$$

$\psi(x) \leftarrow$ or $\psi_n(x)$ since many eigenstates

For ∞ square well $\psi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$

For Harmonic oscillator $\psi(x) \sim$ Hermite polys $\times\ e^{-x^2\alpha}$

etc.

Or, even simpler, can ask what linear combinations of $|x\rangle$

give eigenstates of $\hat{p}$, the momentum operator. What

is answer?

$$\int dx\ e^{ipx/\hbar} |x\rangle$$

Similar problem/goal in adding A.M.

We have eigenstates of $L_1^2, L_2^2, L_1^z, L_2^z$

We ask: What linear combinations of these

are eigenstates of $(L_1+L_2)^2$ and $L_1^z+L_2^z$?

In a way much simpler because finite (small)

dim Hilbert space involved.

H He
Li Be B C N O F Ne

Adding $l_1 = 1$ and $l_2 = 1/2$ is HW problem

what element?

could arise as e^- in p shell $\underbrace{1s^2 2s^2}_{\text{closed shell } J=0} 2p^1$

what is j here

$l_1\ l_2\ m_1\ m_2$

$|1\ 1/2\ 1\ 1/2\rangle$
$0\ 1/2\rangle$
$-1\ 1/2\rangle$
$1\ -1/2\rangle$
$0\ -1/2\rangle$
$-1\ -1/2\rangle$

6 dim Hilbert space

$1 + 1/2 = 3/2,\ 1/2$

$4 + 2$

Here we will do $l_1 = 1$ $l_2 = 1$

Hilbert space dimension?

$\lvert 1\ 1\ 1\ 1\rangle \to$	$\lvert + +\rangle$	$\lvert + 0\rangle$	$\lvert + -\rangle$
$0\ 1\rangle \to$	$\lvert 0 +\rangle$	$\lvert 0\ 0\rangle$	$\lvert 0 -\rangle$
$-1\ 1\rangle \to$	$\lvert - +\rangle$	$\lvert -\ 0\rangle$	$\lvert - -\rangle$

$$(L_1 + L_2)^2 |++\rangle = (L_1^2 + 2L_1 \cdot L_2 + L_2^2)|++\rangle$$

$$L_1^+ L_2^- + L_1^- L_2^+ + 2L_1^z L_2^z$$

$$= 2\hbar^2|++\rangle + 0 + 0 + 2\hbar^2|++\rangle + 2\hbar^2|++\rangle$$

$$= 6\hbar^2|++\rangle \qquad l = 2 \qquad (l(l+1)\hbar^2 = 6\hbar^2)$$

Claim: To get another $\ell=2$ combination, act with

$L_1^- + L_2^-$ on $|++\rangle$

Recall
$$L^+|\ell m\rangle = \hbar\sqrt{\ell(\ell+1)-m(m+1)}\,|\ell\, m+1\rangle$$
$$L^-|\ell m\rangle = \hbar\sqrt{\ell(\ell+1)-m(m-1)}\,|\ell\, m-1\rangle$$

$$(L_1^- + L_2^-)|++\rangle = \sqrt{2}\hbar\left(|+0\rangle + |0+\rangle\right)$$

Normalize $\quad 1/\sqrt{2}\left(|+0\rangle + |0+\rangle\right)$

$$(L_1+L_2)^2|+0\rangle = \left(L_1^2 + L_1^+L_2^- + L_1^-L_2^+ + 2L_1^zL_2^z + L_2^2\right)|+0\rangle$$

$$= 2\hbar^2|+0\rangle + 0 + 2\hbar^2|0+\rangle + 0 + 2\hbar^2|+0\rangle$$

$$(L_1+L_2)^2|+0\rangle = 4\hbar^2|+0\rangle + 2\hbar^2|0+\rangle$$

clearly $(L_1+L_2)^2|0+\rangle = 2\hbar^2|+0\rangle + 4\hbar^2|0+\rangle$

$$(L_1+L_2)^2\left(|+0\rangle + |0+\rangle\right) = 6\hbar^2\left(|+0\rangle + |0+\rangle\right) \quad \checkmark$$

Do it again! $\quad (L_1^- + L_2^-)\left(|+0\rangle + |0+\rangle\right)$

$$= \sqrt{2}\hbar|00\rangle + \sqrt{2}\hbar|+-\rangle + \sqrt{2}\hbar|-+\rangle + \sqrt{2}\hbar|00\rangle$$

$$= \sqrt{2}\hbar\left\{|+-\rangle + 2|00\rangle + |-+\rangle\right\}$$

Check it out:

$$(L_1+L_2)^2\,|+-\rangle = \left(L_1^2 + L_1^+L_2^- + L_2^+L_1^- + 2L_1^zL_2^z + L_2^2\right)|+-\rangle$$

$$= 2\hbar^2|+-\rangle + 0 + 2\hbar^2|00\rangle - 2\hbar^2|+-\rangle + 2\hbar^2|+-\rangle$$

$$(L_1+L_2)^2|+-\rangle = 2\hbar^2|+-\rangle + 2\hbar^2|00\rangle$$

$$(L_1+L_2)^2|-+\rangle = 2\hbar^2|-+\rangle + 2\hbar^2|00\rangle$$

$$(L_1+L_2)^2|00\rangle = 2\hbar^2|00\rangle + 2\hbar^2|+-\rangle + 2\hbar^2|-+\rangle + 0 + 2\hbar^2|00\rangle$$

$$(L_1+L_2)^2\left(|+-\rangle + 2|00\rangle + |-+\rangle\right)$$

$$\begin{aligned} = {} & \hbar^2\left[\,2 + 4\,\right]|+-\rangle \\ & + \hbar^2\left[\,2 + 8 + 2\,\right]|00\rangle \\ & + \hbar^2\left[\,+4 + 2\,\right]|-+\rangle \end{aligned}$$

$$= 6\hbar^2\left(|+-\rangle + 2|00\rangle + |-+\rangle\right) \qquad \checkmark$$

So far

Guess

$$\left.\begin{array}{lc} & M \\ |++\rangle & +2 \\ \frac{1}{\sqrt{2}}\left(|+0\rangle + |0+\rangle\right) & +1 \\ \frac{1}{\sqrt{6}}\left(|+-\rangle + 2|00\rangle + |-+\rangle\right) & 0 \\ \frac{1}{\sqrt{2}}\left(|-0\rangle + |0-\rangle\right) & -1 \\ |--\rangle & -2 \end{array}\right\}\ \ell = 2$$

Check it out!

$$(L_1^- + L_2^-)\left(|+-\rangle + 2|00\rangle + |-+\rangle\right)$$

$$= \sqrt{2}\hbar\left(|0-\rangle + 2|-0\rangle + 2|0-\rangle + |-0\rangle\right)$$

$$\sim \left(|0-\rangle + |-0\rangle\right)$$

Now there is $l=1$ $m=1$

Must build out of $|+0\rangle$ and $|0+\rangle$ and must be $\perp$

to $l=2$ $m=1$. Only possibility is

$$\frac{1}{\sqrt{2}}\left(|+0\rangle - |0+\rangle\right)$$

Try it!

$$(L_1+L_2)^2\,|+0\rangle = 4\hbar^2|+0\rangle + 2\hbar^2|0+\rangle$$ see page AM-23

$$(L_1+L_2)^2\,|0+\rangle = 2\hbar^2|+0\rangle + 4\hbar^2|0+\rangle$$

$$(L_1+L_2)^2\left(|+0\rangle - |0+\rangle\right) = 2\hbar^2\left(|+0\rangle - |0+\rangle\right)$$

So eigenstate $l(l+1) = 2 \implies l=1$ ✓✓

Continue process

$(L_1^- + L_2^-)$ 2 more times to get $m=0$ and $m=-1$

of the $l=1$ result. Finally get $l=0$ $m=0$ by

orthogonality to $l=2$ $m=0$ and $l=1$ $m=0$

$(L_1^- + L_2^-)(|+0\rangle - |0+\rangle)$

$= |00\rangle + |+-\rangle - |-+\rangle - |00\rangle$

$= |+-\rangle - |-+\rangle$ ← NB $\perp$ to $l=2$ $m=0$!

$|+-\rangle + 2|00\rangle + |-+\rangle$

$|+-\rangle \qquad\qquad - |-+\rangle$

$a|+-\rangle + b|00\rangle + c|-+\rangle$

$\perp$: $a + 2b + c = 0$

$a - c = 0$ ← $a = c$

$2a + 2b = 0$ $\quad a = -b$

$c = -b$

$\Rightarrow \frac{1}{\sqrt{3}}(|+-\rangle - |00\rangle + |+-\rangle)$ must be $l=0$

$m=0$