Before doing H atom:
KEPLER – 1/r potential in classical Mechanics

Will not do full complexity but review "easy" stuff

Focus on role of angular momentum, since that is key in QM also. Simplicity: M_1 pinned at origin (can go to relative coordinates)

$$\vec{F}_2 = GM_1M_2/r^2\,\hat{r} \implies \vec{a}_2 = GM_1/r^2\,\hat{r}$$

consider $\vec{L} = \vec{r} \times \vec{p}$

$$d\vec{L}/dt = \underbrace{\vec{v} \times \vec{p}}_{0} + \underbrace{\vec{r} \times m\vec{a}_2}_{0}$$

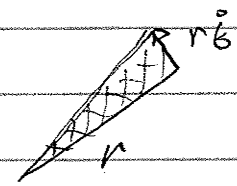
$\vec{L}$ = constant $\implies$ motion is in plane $\perp$ to $\vec{L}$
(true for any central force!)

Use planar polar coordinates

$$\vec{r} = r\hat{r} \qquad \vec{v} = \frac{dr}{dt}\hat{r} + r\dot{\theta}\,\hat{\theta}$$

$$\vec{L} = \vec{r} \times m_2\vec{v} = m_2 r^2\dot{\theta}\,\hat{z}$$ ← unit vector ⊥ to plane

$|\vec{L}| = \ell = m_2 r^2 \dot{\theta}$ is constant ← (One of Kepler's laws!)



$$\frac{dA}{dt} = \text{constant!}$$

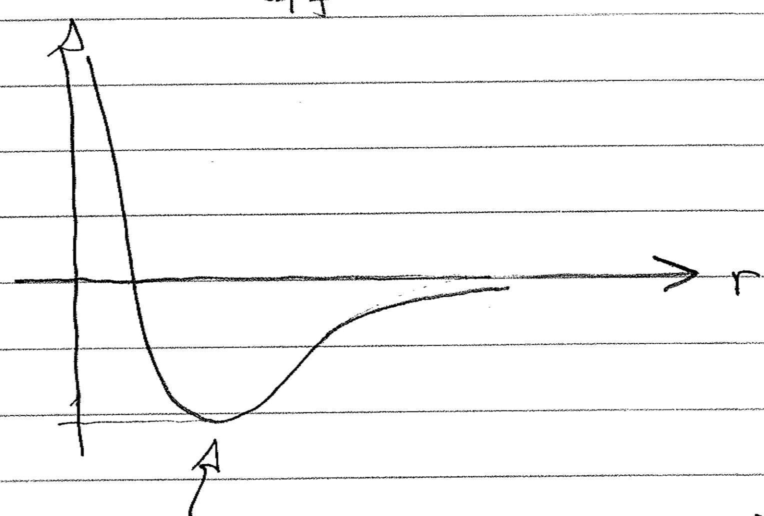
RM - 2

$$E = -\frac{GM_1M_2}{r} + \underbrace{\tfrac{1}{2}m_2v^2}_{\frac{1}{2}m_2(\dot{r}^2 + r^2\dot{\theta}^2)}$$

constant

$$E = -\frac{GM_1M_2}{r} + \frac{1}{2}m_2\dot{r}^2 + \frac{1}{2}m_2 r^2\left(\frac{\ell}{m_2 r^2}\right)^2$$

$$\frac{1}{2}m_2\dot{r}^2 + \underbrace{\frac{\ell^2}{2m_2 r^2} - \frac{GM_1M_2}{r}}_{V_{eff}} = E$$



minimum @ $\frac{dV_{eff}}{dr} = 0 = -\frac{\ell^2}{m_2 r^3} + \frac{GM_1M_2}{r^2}$

$$\frac{M_2^{\cancel{2}}\, r^4\dot{\theta}^2}{\cancel{M_2}\, r^3} = \frac{GM_1M_2}{r^2}$$

$$M_2\frac{(r\dot{\theta})^2}{r} = \frac{GM_1M_2}{r^2}$$

usual circular orbit condition! $\longrightarrow$ $m_2\frac{V_\theta^2}{r} = \frac{GM_1M_2}{r^2}$

KE at r_{min} given by

$$\frac{1}{2} M_2 V_0^2 = \frac{1}{2} M_2 \frac{G M_1}{r} = -\frac{1}{2} PE$$

so $\quad PE + KE < 0 \quad$ (Virial Theorem)

negative (2× KE)

positive

A bit more $\quad E \neq E_{min} \quad$ can deduce turning points