Identical Particles

In classical Mechanics, can treat particles completely independently if they do not interact

$$V(\vec{r}_1, \vec{r}_2) = V_a(\vec{r}_1) + V_b(r_2)$$

$$\vec{F}_1 = -\nabla_1 V(\vec{r}_1, \vec{r}_2) = -\nabla_1 V_a$$

$$\vec{F}_2 = -\vec{\nabla}_2 V(r_1, r_2) = -\nabla_2 V_b$$

$$m\ddot{\vec{r}}_1 = \vec{F}_1$$

same eqn as if ② did not exist!

$$m\vec{r}_2 = \vec{F}_2$$

It is not quite so simple in QM. Even without interactions, quantum particles affect each other. How?

$\boxed{\text{PAULI}}$ We can understand periodic table from H atom (no interactions) but $\boxed{\text{only}}$ if we assume Pauli.

Ie Periodic table requires this! Otherwise, a Carbon atom could have all its electrons in $R_1(r) Y_0^0(\theta, \phi)$

$$-\frac{\hbar^2}{2m_1}\nabla_1^2\psi - \frac{\hbar^2}{2m_2}\nabla_2^2\psi + V(r_1,r_2)\psi = E\psi$$

$$V_a(r_1) + V_b(r_2)$$

~~Try~~ $\cancel{\psi(r_1 r_2) = \psi_a(r_1)\,\psi_b(r_2)}$

Solve indiv ~~Sch Eqn~~:

$$\left[-\frac{\hbar^2}{2m_1}\nabla_1^2 + V_a\right]\psi_a(r_1) = E_a\psi_a(r_1)$$

$$\left[-\frac{\hbar^2}{2m_2}\nabla_2^2 + V_b\right]\psi_b(r_2) = E_b\psi_b(r_2)$$

And Try $\psi(r_1 r_2) = \psi_a(r_1)\,\psi_b(r_2)$

Does it work? $\boxed{\text{YES}}$

But if particles are identical. Doesn't really make sense.

Easy to fix

Don't worry about normalization right now

$$\psi_\pm(r_1 r_2) \propto \psi_a(r_1)\psi_b(r_2) \pm \psi_a(r_2)\psi_b(r_1)$$

+ bosons ← integer spin

− fermion ← half integer spin

If $a=b$ (Bosons) →

$$\psi_+(r_1 r_2) \propto \psi_a(r_1)\psi_a(r_2)$$

$$\psi_-(r_1 r_2) = 0 \longrightarrow \text{PAULI}$$

(Fermion)

Notice, as a consequence

$$\psi(r_1\, r_2) = -\psi(r_2\, r_1) \qquad \text{fermions} \qquad \text{antisymmetric}$$

$$\psi(r_1\, r_2) = \psi(r_2\, r_1) \qquad \text{bosons} \qquad \text{symmetric}$$

Actually this is fundamental principle <u>leading</u> to choice

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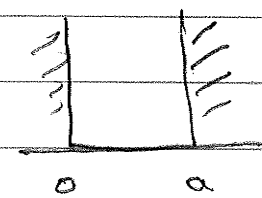
Identical particles in 1d ∞ square well



$$\psi_n = \sqrt{\frac{2}{\pi}}\, \sin \frac{n\pi x}{a}$$

$$E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2 = \frac{\pi^2 \hbar^2}{2ma^2}\, n^2 \equiv k n^2$$

<u>Distinguishable</u> <u>particles</u> (eg an electron and a proton)

ground state $\psi(x_1 x_2) = \frac{2}{a} \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{a}$ $\qquad E_{11} = 2k$

excited states $\psi(x_1, x_2) = \frac{2}{a} \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{a}$ $\qquad E_{12} = 5k$

$\frac{2}{a} \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{a}$ $\qquad E_{21} = 5k$

Bosons : same ground state. (Note $\psi_a(r_1)\psi_b(r_2)+\psi_a(r_2)\psi_b(r_1)$
$= 2\psi_a(r_1)\psi_a(r_2)$

if $a=b$)

$E = 2k$

But only one first excited state

$\psi_+(x_1,x_2)=\frac{\sqrt{2}}{a}\left(\sin\frac{\pi x_1}{a}\sin\frac{2\pi x_2}{a}+\sin\frac{2\pi x_1}{a}\sin\frac{\pi x_2}{a}\right)$

$\frac{2}{a^2}\left(\frac{a}{2}\ \frac{a}{2}+\frac{a}{2}\ \frac{a}{2}\right)$ ✓✓ why no cross terms?

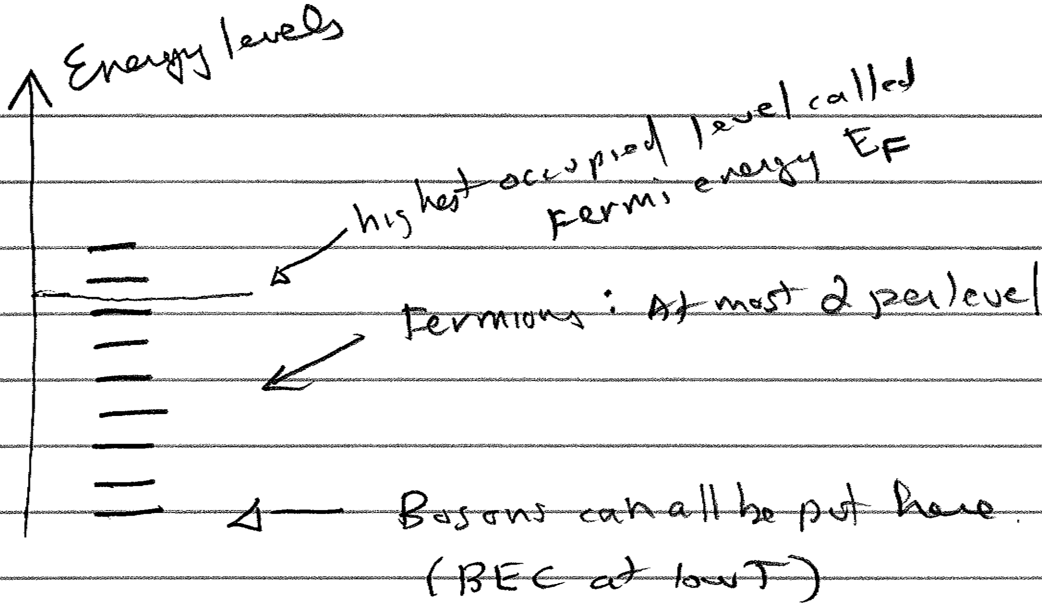
$E=5k$

Fermions Ground state is $E=5k$

$\psi_-(x_1,x_2)=\frac{\sqrt{2}}{a}\left(\sin\frac{\pi x_1}{a}\sin\frac{2\pi x_2}{a}-\sin\frac{2\pi x_1}{a}\sin\frac{\pi x_2}{a}\right)$

This is basis of idea behind high KE of e^- in metal!

ground state energy 5k fermions >> 2k bosons



What would wave function be

$$\psi(x_1 x_2) = \psi_a(x_1)\psi_b(x_2) - \psi_a(x_2)\psi_b(x_1)$$

$$= \begin{vmatrix} \psi_a(x_1) & \psi_b(x_1) \\ \psi_a(x_2) & \psi_b(x_2) \end{vmatrix}$$ "slater determinant"

$$\psi(x_1 x_2 x_3) = \begin{vmatrix} \psi_a(x_1) & \psi_b(x_1) & \psi_c(x_1) \\ \psi_a(x_2) & \psi_b(x_2) & \psi_c(x_2) \\ \psi_a(x_3) & \psi_b(x_3) & \psi_c(x_3) \end{vmatrix}$$

$$= \psi_a(x_1)\psi_b(x_2)\psi_c(x_3) - \psi_a(x_1)\psi_b(x_3)\psi_c(x_2)$$

$$+ \ldots$$

Minus sign upon $x_2 \leftrightarrow x_3$

Why mathematically?

e^- in metal can be treated as non interacting

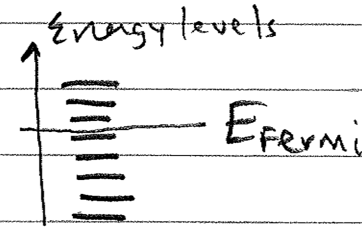
because they are at such high density $\Uparrow\Uparrow$

!!?? counterintuitive!

Typical density of e^- in metal $\sim 10^{22}/cm^3$

Energy levels $\frac{\hbar^2 k^2}{2m}$ but can put at most $2e^-$ in each (spin)

$E_0 \neq 0$ ie cannot put all in $k=0$



$$N = \underbrace{2}_{\text{spin}} \frac{V}{(2\pi)^3} \int d^3k$$

$$= \frac{V}{2\pi^3} 4\pi \int_0^{k_F} k^2 \, dk$$

Seen this?
One motivation is dimensional

$$= \frac{V}{\pi^2} \frac{k_F^3}{3}$$

$$\rho = \frac{N}{V} = \frac{k_F^3}{3\pi^2} \qquad k_F = (3\pi^2 \rho)^{1/3}$$

$$KE \Leftrightarrow E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}(3\pi^2\rho)^{2/3} \sim \rho^{2/3}$$

~~KE~~ Can show $\langle KE \rangle = \frac{3}{5} E_F$

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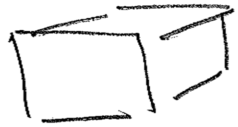
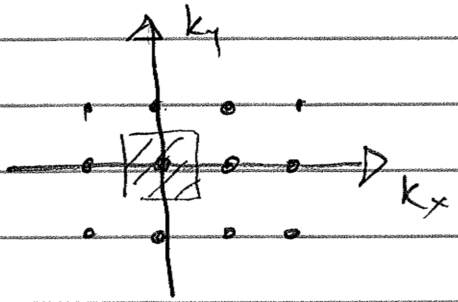


$\psi_k = e^{i\vec{k}\cdot\vec{r}}$

+ PBC

$$k_x = \frac{2\pi n_x}{L} \qquad k_y = \frac{2\pi n_y}{L} \qquad k_z = \frac{2\pi n_z}{L}$$

$n_x \; n_y \; n_z$ integers



Volume per k point is

$$\left(\frac{2\pi}{L}\right)^3$$

2D picture

$\therefore$ # points in k space

is $$\frac{1}{(2\pi/L)^3}\int d^3k$$

$$= \frac{L^3}{(2\pi)^3}\int d^3k = \frac{V}{(2\pi)^3}\int d^3k .$$

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What about PE?

PE ~