Heisenberg Model

This work on adding angular momentum is closely connected to research in magnetism: The "Heisenberg Hamiltonian"

$$\hat{H} = +J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$

$\vec{S}_i$ are spin 1/2 operators on sites of lattice

Consider just two sites

classical analog E of 2 current loops $E \propto -\frac{\vec{\mu}_1 \cdot \vec{\mu}_2}{r^3}$

$$\hat{H} = +\frac{J}{\hbar^2} \vec{S}_1 \cdot \vec{S}_2$$

$$= +\frac{J}{\hbar^2}\left(S_{1x} S_{1y} + S_{2x} S_{2y} + S_{1z} S_{2z}\right)$$

$$= +\frac{J/2}{\hbar^2}\left(S_{1+} S_{2-} + S_{1-} S_{2+}\right) + \frac{J}{\hbar^2} S_{1z} S_{2z}$$

$$\hat{H}|++\rangle = +J/4\,|++\rangle$$

$$\hat{H}|--\rangle = +J/4\,|--\rangle$$

$$\hat{H}|+-\rangle = +\frac{J}{2}|-+\rangle - \frac{J}{4}|+-\rangle$$

$$\hat{H}|-+\rangle = +\frac{J}{2}|-+\rangle - \frac{J}{4}|-+\rangle$$

we are just redoing calculation of adding 2 spin-1/2's!

$$\begin{vmatrix} -J/4 & +J/2 \\ +J/2 & -J/4 \end{vmatrix} \rightarrow \left(-J/4 - \lambda\right)^2 - J^2/4 = 0$$

$$-J/4 - \lambda = \pm J/2$$

$$\lambda = J/4 \qquad \lambda = -3J/4$$

Singlet $\frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$ is lowest energy $-3J/4$

Another way:

$$\hat{H} = \frac{J}{\hbar^2}\left(\hat{S}_1 \cdot \vec{S}_2\right)$$

$$= \frac{J}{2\hbar^2}\left[\underbrace{(S_1+S_2)^2}_{0,\,1} - \underbrace{S_1^2}_{1/2} - \underbrace{S_2^2}_{1/2}\right]$$

singlet $\quad 0(0+1) - \frac{1}{2}\left(\frac{3}{2}\right)\hbar^2 - \frac{1}{2}\left(\frac{3}{2}\right)\hbar^2 \longrightarrow -\frac{J}{2}\left(\frac{3}{2}\right)$

$$= -3J/4$$

triplet