GAUGE TRANSFORMATIONS IN QM

given a wave function $\psi(\vec{r})$ we can multiply by any phase and not change physics

$$\psi(\vec{r}) \to e^{i\theta}\,\psi(\vec{r})$$

$$\langle \hat{O} \rangle_{\theta \neq \phi} = \int d^3r\, \psi^*(\vec{r})\, e^{-i\theta}\, \hat{O}\, e^{i\theta}\, \psi(\vec{r}) = \langle \hat{O} \rangle_{\theta = \phi}$$

any observable

pass through

Suppose $\theta = \theta(\vec{r})$ instead of being constant.

"local gauge transformation"

"global gauge transformation"

Then we cannot pass $e^{+i\theta(\vec{r})}$ through $\hat{O}$ eg if it has any momentum operators (derivatives).

$$\hat{p}\left(e^{i\theta(\vec{r})}\psi\right) = \frac{\hbar}{i}\nabla\left(e^{i\theta(\vec{r})}\psi\right)$$

$$= \hbar\nabla\theta\, e^{i\theta(r)}\psi + \frac{\hbar}{i}e^{i\theta(r)}\nabla\psi$$

extra term

We would get $\langle \hat{O} \rangle_{\theta \neq 0} = \langle \hat{O} \rangle_{\theta = 0}$

If we insist that whenever we do local gauge transformation $\psi(\vec{r}) \to e^{i\theta(\vec{r})} \psi(\vec{r})$

at the same time we change

$$\vec{p} \to \vec{p} + \hbar \nabla \theta$$

to cancel out the extra term

Putting together with $\vec{p} \to \vec{p} - \frac{e}{c}\vec{A}$

where $\vec{A}$ is invariant to with $\vec{\nabla}\Lambda$ we see

that QM is gauge invariant if whenever we change gauge $\vec{A} \to \vec{A} + \vec{\nabla}\Lambda$ we also transform the wavefunction locally via

$$\psi(\vec{r}) \to e^{-\frac{ie}{c\hbar}\Lambda(r)} \psi(\vec{r})$$

$+\hbar \vec{\nabla}\theta \leftrightarrow -\frac{e}{c}\vec{\nabla}\Lambda$

$\theta \leftrightarrow -\frac{e}{\hbar c}\Lambda$

The abstract (and amazingly beautiful!)

way of looking at this calculation is:

If you insist on QM being locally gauge invariant **then** you are **forced** to postulate the existence of magnetic fields!!

An example of how symmetry $\Rightarrow$ constrains possible theories

Much more complex reasoning nowadays

If ⊛ $\Rightarrow$ Higgs particle must exist etc.