

Diffusion Eqn

$$D \nabla^2 \psi(\vec{r}, t) = \frac{\partial \psi(\vec{r}, t)}{\partial t}$$

* Conservation law proof *

With a source term

$$[D \nabla^2 + \lambda] \psi(\vec{r}, t) = \frac{\partial \psi(\vec{r}, t)}{\partial t}$$

To understand
Physics of why

$$\frac{\partial \rho}{\partial t} \sim \frac{\partial^2 \rho}{\partial x^2}$$

consider $\rho = c$

$$\rho = a + bx$$

and ask if we expect $\frac{\partial \rho}{\partial t} \neq 0$

Separate variables: $\psi(\vec{r}, t) = u(\vec{r}) g(t)$

$$g [D \nabla^2 + \lambda] u = u \frac{dg}{dt}$$

$$\frac{1}{u} [D \nabla^2 + \lambda] u = -\alpha = \frac{1}{g} \frac{dg}{dt}$$

$$g(t) = e^{-\alpha t}$$

← How does this compare
with Schrodinger?

$$[D \nabla^2 + \lambda + \alpha] u(\vec{r}) = 0$$

In a cubic box $u(\vec{r}) = \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{a} \sin \frac{l\pi z}{a}$

$$D \frac{\partial^2 u}{\partial x^2} - D \frac{\pi^2}{a^2} (n^2 + m^2 + l^2) + \lambda + \alpha = 0$$

$$\alpha = -\lambda + \frac{D\pi^2}{a^2} (n^2 + m^2 + l^2)$$

↑
okay for D large or a small
 α will be positive But if $\alpha < 0$
exponential growth

DE-2

Most dangerous mode is $n=m=l=1$ when λ term smallest

$$\lambda = 3D\pi^2/a^2$$

Critical size $a^2 = 3D\pi^2/\lambda$

Critical volume $a^3 = \left(3D\pi^2/\lambda\right)^{3/2}$

Consider now a spherical geometry

$$[D \nabla^2 + \lambda + \alpha] u(\vec{r}) = 0$$

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V(r)\right] \psi(\vec{r}) = E \psi(r)$$

Thus $D \leftrightarrow -\frac{\hbar^2}{2m} \quad (\lambda + \alpha) \leftrightarrow V - E$

$\therefore u(\vec{r}) = R(r) Y_{lm}(\theta, \phi)$ by analogy, ...

where $\frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{r^2}{D} [\lambda + \alpha] R = l(l+1) R$

Or, in terms of $u(r) = r R(r)$

$$D \frac{d^2 u}{dr^2} + \left[\lambda - D \frac{l(l+1)}{r^2} \right] u = -\alpha u$$

From Sch. Discussion $\frac{d^2 u}{dr^2} = \left(\frac{l(l+1)}{r^2} - k^2 \right) u$ has soln $j_l(kr)$

Here $\frac{d^2 u}{dr^2} = \left[\frac{l(l+1)}{r^2} - \frac{\lambda + \alpha}{D} \right] u$

Solns are $j_e \left(\sqrt{\frac{\lambda + \alpha}{D}} r \right)$

Condition that $R \rightarrow 0$
 $u \rightarrow 0$ at $r = a$ requires

$$\sqrt{\frac{\lambda + \alpha}{D}} a = \beta_{ne} \leftarrow n^{\text{th}} \text{ root of } j_e$$

solving for α yields

$$(\lambda + \alpha) a^2 = D \beta_{ne}^2$$

$$\cancel{a^2} = \cancel{D \beta_{ne}^2} \quad \alpha = -\lambda + \frac{D \beta_{ne}^2}{a^2}$$

As before we want $\alpha > 0$ to avoid "explosion"

Most dangerous mode has smallest β_{ne} . This is $l=0$

where $\beta_{10} = \pi$ (see Griffiths p 143)

$$\cancel{a^2} = \cancel{D \pi^2} \quad a^2 = \frac{D \pi^2}{\lambda}$$

$$a^3 = \left(\frac{D \pi^2}{\lambda} \right)^{3/2} \quad \frac{4}{3} \pi a^3 = \frac{4}{3} \pi \left(\frac{D \pi^2}{\lambda} \right)^{3/2}$$

Compare to cube $V_{\text{sphere}}^{\text{crit}} = \frac{4}{3} \pi \left(\frac{D \pi^2}{\lambda} \right)^{3/2}$

$$V_{\text{cube}}^{\text{crit}} = 3^{3/2} \left(\frac{D \pi^2}{\lambda} \right)^{3/2}$$

as expected

$$V_{\text{sphere}}^{\text{crit}} < V_{\text{cube}}^{\text{crit}}$$

4.19

5.20