SOLUTIONS

MIDTERM EXAM

Physics 115B– FALL 2011

Analytic:

[1.] How does the spherical harmonic $Y_{lm}(\theta, \phi)$ depend on the azimuthal angle ϕ?

[2.] Consider a charged particle in a one dimensional harmonic oscillator potential. Suppose we turn on a weak electric field (E) so that the potential is shifted an amount $H' = -qEx$. Show that there is no first order shift in the energy levels and calculate the second order correction. Note that $x = \sqrt{\frac{\hbar}{2m\omega}}\,(a\ + a^\dagger\,)$ in terms of the raising and lowering operators.

[3.] Compute $\langle r \rangle$ and $\langle r^2 \rangle$ for the ground state of the Hydrogen atom $\Psi_{100} = e^{-r/a}/\sqrt{\pi a^3}$. Find also the most probable value of r.

[4.] In certain respects, spherical Bessel functions look like sine and cosine: they oscillate positive and negative as their argument increases. However, the Bessel oscillations are not as simple as those of sine and cosine. In what two significant ways is the behavior more complicated?

[5.] The solution of the radial Schroedinger equation for the Hydrogen atom involves,

$$\frac{d^2u(\rho)}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] u(\rho)$$

By dropping appropriate terms, derive the asymptotic forms of $u(\rho)$ as $\rho \to \infty$, and as $\rho \to 0$.

Numeric:

[6.] Write a loop (in C or C++) which will implement Molecular Dynamics (MD) for the classical harmonic oscillator. That is, your loop should evolve position x and velocity v forward N time steps of length dt. You do not need to write other parts of the code (input/output, declaration of variables, etc.) As much as you might like to write glorious, fancy code, please limit your response to ten lines. You will be graded both on getting the MD algorithm correct and also on correct syntax, so include all the required brackets, semicolons, etc that would be needed for correct compilation.

Physics 115B - MIDTERM SOLUTIONS 11/2/2011

① $Y_\ell^m \propto e^{im\phi}$

② Bessel functions (compared with sine/cosine functions):

1) have zeros that are irregularly spaced on x-axis
   or (equivalent) frequency changes along x-axis.
2) have amplitudes that decay from the origin

③ $H = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2 \qquad H' = -qEx$

Eigenvalues $= \hbar\omega(n+1/2)$ Eigenfunctions: $\{|n\rangle\}$

1st order: $E_n^1 = \langle n|H'|n\rangle = -qE\sqrt{\frac{\hbar}{2m\omega}}\langle n|a^\dagger + a|n\rangle = 0$ where $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)$

2nd order: $E_n^2 = \sum_{m\neq n}' \frac{|\langle m|H'|n\rangle|^2}{E_n^0 - E_m^0} = \frac{q^2E^2\hbar}{2m\omega}\sum_{m\neq n}' \frac{|\langle m|a^\dagger + a|n\rangle|^2}{E_n^0 - E_m^0}$

Non-zero only for $m = n+1, n-1$: $E_n^2 = \frac{q^2E^2\hbar}{2m\omega}\left(\frac{n+1}{-\hbar\omega} + \frac{n}{+\hbar\omega}\right) = \frac{q^2E^2}{2m\omega^2}(-1) = \boxed{\frac{-q^2E^2}{2m\omega^2}}$

④ $\langle r\rangle = \int \psi_{100}^* r\, \psi_{100} = \frac{1}{\pi a^3}\int e^{-2r/a} r^3 dr\,(4\pi) = \frac{4\pi}{\pi a^3}\int_0^\infty e^{-2r/a} r^3 dr = \frac{4(3!)}{a^3}\left(\frac{a}{2}\right)^4$

$\langle r\rangle = \boxed{3/2\, a}$

$\langle r^2\rangle = \frac{4\pi}{\pi a^3}\int e^{-2r/a} r^4 = \frac{4}{a^3}(4!)\left(\frac{a}{2}\right)^5 = \frac{\cancel{2^4}\cdot 3 \cdot \cancel{2}\, a^5}{a^3\, \cancel{2^5}} = \boxed{3a^2}$

$P(r < r < r+dr) = |\psi_{100}|^2 r^2 \sin\theta = A e^{-2r/a} r^2 \sin\theta$

$\frac{\partial P(r)}{\partial r} = A\ldots\left[-\frac{2}{a}e^{-2r/a} r^2 + 2r e^{-2r/a}\right] = 0$

$A\,2r e^{-2r/a}\left[-\frac{1}{a}r + 1\right] = 0 \Rightarrow \boxed{r = a}$ most probable

⑤ $\frac{d^2u(\rho)}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{\ell(\ell+1)}{\rho^2}\right]u(\rho)$

$\rho \to \infty \Rightarrow \frac{d^2u(\rho)}{d\rho^2} = u(\rho) \Rightarrow u(\rho) = Ae^{\rho} + Be^{-\rho} \Rightarrow \boxed{u(\rho) = Ae^{-\rho}}$

$\rho \to 0 \Rightarrow \frac{d^2u(\rho)}{d\rho^2} = \frac{\ell(\ell+1)}{\rho^2}u(\rho) \Rightarrow u(\rho) = A\rho^{\ell+1} + B\rho^{-\ell} \Rightarrow \boxed{u(\rho) = A\rho^{\ell+1}}$

$F = -kx$

$a = -\frac{k}{m} x$

#6

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for (int i=0; i<N; i++) {
        x = x + vx * dt;
        vx = vx - (k/m) * x * dt;
    }
```