SOLUTIONS

PROBLEM SET 6 Due Wednesday November 23

Physics 115B– FALL 2011

Pts Assigned Analytic:

10 [**1.**] Griffiths Problem 4.60

10 [**2.**] Griffiths Problem 4.61

10 [**3.**] Griffiths Problem 5.1

10 [**4.**] Griffiths Problem 5.2

10 [**5.**] Solve for the energy levels of the four site Heisenberg model with a J' term:

$$H = J\Big(S_1 \cdot S_2 + S_2 \cdot S_3 + S_3 \cdot S_4 + S_4 \cdot S_1 \Big) + J'\Big(S_1 \cdot S_3 + S_2 \cdot S_4 \Big)$$

Hint: We solved the $J' = 0$ case in class. A very similar procedure works here. You may need to combine an idea from the two site problem to express H purely as squares of operators. Show that as you increase J' up from zero there is a "critical point" $J'_c = \alpha J$ where the lowest energy state (ground state) changes. What is the value of α? This same thing change in behavior occurs for larger systems, in which case it is referred to as a "quantum phase transition." In a classical phase transition there is a sharp change in the nature of a system when the temperature goes through a critical value T_c. Analogously, in a "quantum phase transition" there is a sharp change in the ground state of a quantum system when one of the parameters in the Hamiltonian changes.

P115B HW7 SOLUTIONS

#1 Griffiths Problem 4.60

$$\bar{A} = \frac{B_0}{2}(x\hat{y} - y\hat{x}), \quad \phi = Kz^2$$

(a) $$\bar{B} = \nabla \times A = \frac{B_0}{2}\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial x & \partial y & \partial z \\ -y & x & 0 \end{pmatrix} = \frac{B_0}{2}\left(0 + 0 + (2)\hat{z}\right) = \boxed{B_0\hat{z}}$$

$$\bar{E} = -\nabla\phi - \frac{\partial\bar{A}}{\partial t} = -2Kz\hat{z} + 0 = \boxed{-2Kz\hat{z}}$$

(b) SE: $$\frac{1}{2m}\left(\hat{p} - \frac{q}{c}A\right)^2\psi + q\phi\psi = E\psi$$

$$\frac{1}{2m}\left(\hat{p} - \frac{q}{c}A\right)^2 = \frac{1}{2m}\left[\left(p_x + \frac{qyB_0}{2c}\right)^2 + \left(p_y - \frac{qxB_0}{2c}\right)^2 + p_z^2\right]$$

$$\frac{1}{2m}\left[p_x^2 + \frac{qB_0}{2c}(p_x y + y p_x) + \left(\frac{qB_0}{2c}\right)^2 y^2 + p_y^2 - \frac{qB_0}{2c}(p_y x + x p_y) + \left(\frac{qB_0}{2c}\right)^2 + p_z^2\right]$$

$$\frac{1}{2m}\left[p_x^2 + p_y^2 + p_z^2 + \left(\frac{qB_0}{2c}\right)^2(x^2+y^2) + \frac{qB_0}{2c}\left([p_x, y] + 2yp_x - [p_y, x] - 2xp_y\right)\right]$$

$$= \frac{1}{2m}\left[\bar{P}^2 + \left(\frac{qB_0}{2c}\right)^2(x^2+y^2) + \frac{qB_0}{c}(yp_x - xp_y)\right]$$

$$= \frac{1}{2m}\bar{P}^2 + \frac{qB_0}{2mc}L_z + \frac{q^2B_0^2}{8mc^2}(x^2+y^2)$$

So full SE is:

$$\left[\frac{1}{2m}\bar{P}^2 + \frac{qB_0}{2mc}L_z + \frac{q^2B_0^2}{8mc^2}(x^2+y^2) + qKz^2\right]\psi = E\psi$$

Now, separate (x,y) and z components

depends on x, y, p_x, p_y — depends on z, p_z

$$\left[\frac{1}{2m}(p_x^2 + p_y^2) + \frac{q^2B_0^2}{8mc^2}(x^2+y^2) + \frac{qB_0}{2mc}(yp_x - xp_y)\right]\psi + \left[\frac{1}{2m}p_z^2 + qKz^2\right]\psi$$

$$= E_{x,y}\psi + E_z\psi$$

For (x, y) equation:

$$\left[\frac{1}{2m}p_x^2 + \frac{q^2 B_0^2}{8mc^2}x^2 + \frac{1}{2m}p_y^2 + \frac{q^2 B_0^2}{8mc^2}y^2 + \frac{qB_0}{2mc}(yp_x - xp_y)\right]\psi = E_{x,y}\psi$$

Now, use creation/destruction operators to redefine x, y:

$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(a_x + a_x^+) \qquad p_x = i\sqrt{\frac{\hbar m\omega}{2}}\,(a_x^+ - a_x)$$

and let $\omega_1 = \frac{qB_0}{2mc} \Rightarrow \frac{1}{2}m\omega_1^2 = \frac{1}{2}\frac{q^2B_0^2}{4mc^2}$

$$\underbrace{\frac{1}{2m}p_x^2 + \frac{1}{2}m\omega_1^2 x^2}_{\hbar\omega_1(a_x^+ a_x + 1/2)} + \underbrace{\frac{1}{2m}p_y^2 + \frac{1}{2}m\omega_1^2 y^2}_{+\,\hbar\omega_1(a_y^+ a_y + \frac{1}{2})} + \frac{qB_0}{2mc}(yp_x - xp_y)\Big]\psi = E_{x,y}\psi$$

$$\hbar\omega_1(a_x^+ a_x + 1/2) + \hbar\omega_1(a_y^+ a_y + \tfrac{1}{2}) + \omega_1(yp_x - xp_y)\Big]\psi = E_{x,y}\psi$$

Now, let $a_r = \frac{1}{\sqrt{2}}(a_x + i a_y) \qquad a_l = \frac{1}{\sqrt{2}}(a_x - i a_y)$

$$\left\{\begin{array}{l} a_r^+ a_r + a_l^+ a_l = a_x^+ a_x + a_y^+ a_y \\ a_r^+ a_r - a_l^+ a_l = \hbar(a_x^+ a_y - a_y^+ a_x) \end{array}\right\} \text{ From class lecture}$$

$$\hookrightarrow = (yp_x - xp_y)$$

<u>Substituting</u>:

$$\hbar\omega_r(a_r^+ a_r + a_l^+ a_l) + \frac{1}{2}\hbar\omega_r + \hbar\omega_r(a_r^+ a_r - a_l^+ a_l)\Big]\psi = E_{x,y}\psi$$

$$\boxed{\hbar\omega_1(a_r^+ a_r + \tfrac{1}{2})\psi = E_r\psi}$$

or $\boxed{E_r = \hbar\omega_r(n_r + \tfrac{1}{2})}$ where $\omega_r \equiv \frac{qB_0}{m}$

For the (z) equation, we get

$$\left[\frac{1}{2m} p_z^2 + q K z^2\right] \psi = E_z \psi$$

which translates to

$$\left[\frac{1}{2m} p_z^2 + \frac{1}{2} m \omega_z^2 z^2\right] \psi = E_z \psi$$ where $\omega_z = \sqrt{\frac{2qK}{m}}$

This is SHO equation in z co-ordinate only,

so $\boxed{E_z = \hbar \omega_z \left(n_z + 1/2 \right)}$

Final Energy $= E_\rho + E_z \Rightarrow$

$$\boxed{E(n_\rho, n_z) = \hbar \omega_\rho (n_\rho + 1/2) + \hbar \omega_z (n_z + 1/2)}$$

#2 Griffiths 4.61

(a) $\phi' \equiv \phi - \frac{\partial \Lambda}{\partial t}$, $\bar{A}' = \bar{A} + \nabla\Lambda$ (Λ is a scalar field)

Let E, B be fields from ϕ, A
E', B' " " ϕ', A'

$$E = -\nabla\phi - \frac{\partial A}{\partial t} \qquad B = \nabla \times A$$

So: $B' = \nabla \times A' = \nabla \times (\bar{A} + \nabla\Lambda) = \nabla \times \bar{A} + \cancelto{0}{\nabla \times (\nabla\Lambda)}$
$= \nabla \times A = \boxed{B}$

$$E' = -\nabla\phi' - \frac{\partial A'}{\partial t} = -\nabla\phi + \frac{\partial}{\partial t}(\nabla\Lambda) - \frac{\partial A}{\partial t} - \frac{\partial}{\partial t}(\nabla\Lambda)$$

$$= -\nabla\phi - \frac{\partial A}{\partial t} = \boxed{E}$$

(b) With $\Psi' = e^{iq\Lambda/\hbar}\Psi$, A', ϕ' show that

Equation (1) $\Rightarrow$ $i\hbar \frac{\partial \Psi'}{\partial t} = \left[\frac{1}{2m}(-i\hbar\nabla - q\bar{A}')^2 + q\phi'\right]\Psi'$ holds

$$[-i\hbar\nabla - qA - q(\nabla\Lambda)]e^{iq\Lambda/\hbar}\Psi = -i\hbar\left[\left(\frac{iq}{\hbar}\right)\nabla\Lambda e^{iq\Lambda/\hbar}\Psi + e^{iq\Lambda/\hbar}\nabla\Psi\right] - qA\Psi' - q\nabla\Lambda\Psi'$$

$$= q(\nabla\Lambda)\Psi' - i\hbar e^{iq\Lambda/\hbar}(\nabla\Psi) - qA\Psi' - q(\nabla\Lambda)\Psi'$$

$$= -i\hbar e^{iq\Lambda/\hbar}(\nabla\Psi) - q\bar{A}\Psi'$$

Apply again

$$[-i\hbar\nabla - qA - q(\nabla\Lambda)]\left[-i\hbar e^{iq\Lambda/\hbar}(\nabla\Psi) - qA\Psi'\right]$$

$$= -\hbar^2\left(\frac{iq}{\hbar}\right)(\nabla\Lambda)e^{iq\Lambda/\hbar}(\nabla\Psi) - \hbar^2 e^{iq\Lambda/\hbar}\nabla^2\Psi$$
$$+ i\hbar q(\nabla\cdot A)\Psi' + i\hbar q\,\bar{A}\cdot(\nabla\Lambda)\left(\frac{iq}{\hbar}\right)\Psi'$$
$$+ iq\hbar(A\cdot\nabla\Psi)e^{iq\Lambda/\hbar} + q^2A^2\Psi' + iq\hbar e^{iq\Lambda/\hbar}(\nabla\Lambda)(\nabla\Psi) + q^2(A\cdot\nabla\Lambda)\Psi'$$

$$= e^{iq\Lambda/\hbar}\Big[\cancel{-i\hbar q(\nabla\Lambda)(\nabla\psi)} - \hbar^2\nabla^2\psi + i\hbar q(\nabla\cdot A)\psi$$
$$\cancel{-q^2\bar{A}\cdot(\nabla\Lambda)\psi} + i\hbar q(A\cdot\nabla\psi) + q^2A^2\psi$$
$$\cancel{+iq\hbar(\nabla\Lambda)(\nabla\psi)} \cancel{+q^2(A\cdot\nabla\Lambda)\psi}\Big]$$

$$= e^{iq\Lambda/\hbar}\left[-\hbar^2\nabla^2\psi + i\hbar q\,(A\cdot\nabla\psi + (\nabla\cdot A)\psi) + q^2A^2\psi\right]$$

$$= e^{iq\Lambda/\hbar}\left[-i\hbar\nabla - q\bar{A}\right]^2\psi$$

From equation 1, we need to show that

$$i\hbar\frac{\partial\psi'}{\partial t} - q\phi'\psi' = e^{iq\Lambda/\hbar}(-i\hbar\nabla + qA)^2\psi$$

$$= \cancel{i\hbar\left(\frac{iq}{\hbar}\right)e^{iq\Lambda/\hbar}\left(\frac{\partial\Lambda}{\partial t}\right)\psi} + i\hbar e^{iq\Lambda/\hbar}\frac{\partial\psi}{\partial t}$$
$$- q\phi e^{iq\Lambda/\hbar}\psi + \cancel{q\left(\frac{\partial\Lambda}{\partial t}\right)e^{iq\Lambda/\hbar}}\psi$$

$$= e^{iq\Lambda/\hbar}\left(i\hbar\frac{\partial\psi}{\partial t} - q\phi\right)\psi$$

But $i\hbar\frac{\partial\psi}{\partial t} - q\phi\psi = \frac{1}{2m}(-i\hbar\nabla + qA)^2\psi$

is valid since ψ solves SE with ϕ, A!

#3 <u>Griffiths Problem # 5.1</u>

$$\bar{R} = \frac{m_1 \bar{r}_1 + m_2 \bar{r}_2}{m_1 + m_2} \qquad \bar{r} = \bar{r}_1 - \bar{r}_2 \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

(a) $$(m_1+m_2)\bar{R} = m_1\bar{r}_1 + m_2\bar{r}_2 = m_1\bar{r}_1 + m_2(\bar{r}_1 - \bar{r}) = (m_1+m_2)\bar{r}_1 - m_2\bar{r}$$

$$\Rightarrow \bar{r}_1 = \bar{R} + \frac{m_2}{m_1+m_2}\bar{r} = \boxed{\bar{R} + \frac{\mu}{m_1}\bar{r}}$$

$$\Rightarrow \bar{r}_2 = \bar{R} - \frac{m_1}{m_1+m_2}\bar{r} = \boxed{\bar{R} - \frac{\mu}{m_2}\bar{r}}$$

Let $\bar{r} = (x, y, z) \qquad \bar{R} = (X, Y, Z)$

$\bar{r}_1 = (x_1, y_1, z_1) \qquad \bar{r}_2 = (x_2, y_2, z_2)$

Now, $X = \left(\frac{m_1}{m_1+m_2}\right)x_1 + \left(\frac{m_2}{m_1+m_2}\right)x_2$ and $x = x_1 - x_2$

$$(\nabla_1)_x = \frac{\partial}{\partial x_1} = \frac{\partial}{\partial X}\frac{\partial X}{\partial x_1} + \frac{\partial}{\partial x}\frac{\partial x}{\partial x_1} = \left(\frac{m_1}{m_1+m_2}\right)\frac{\partial}{\partial X} + (1)\frac{\partial}{\partial x}$$

y, z same $$\boxed{(\nabla_1)_x = \frac{\mu}{m_2}(\nabla_R)_x + (\nabla_r)_x} \Rightarrow \nabla_1 = \frac{\mu}{m_2}\nabla_R + \nabla_r$$

$$(\nabla_2)_x = \frac{\partial}{\partial x_2} = \frac{\partial}{\partial X}\frac{\partial X}{\partial x_2} + \frac{\partial}{\partial x}\frac{\partial x}{\partial x_2} = \left(\frac{m_2}{m_1+m_2}\right)\frac{\partial}{\partial X} + (-1)\frac{\partial}{\partial x}$$

y, z same $$\boxed{(\nabla_2)_x = \frac{\mu}{m_1}(\nabla_R)_x - (\nabla_r)_x} \Rightarrow \nabla_2 = \frac{\mu}{m_1}\nabla_R - \nabla_r$$

(b) <u>SE</u>: $$-\frac{\hbar^2}{2m_1}\nabla_1^2\psi - \frac{\hbar^2}{2m_2}\nabla_2^2\psi + V(r_1, r_2)\psi = E\psi$$

$$\nabla_1^2\psi = \left(\frac{\mu}{m_2}\nabla_R + \nabla_r\right)^2\psi = \left[\left(\frac{\mu}{m_2}\right)^2\nabla_R^2 + \frac{2\mu}{m_2}\nabla_R\cdot\nabla_r + \nabla_r^2\right]\psi$$

$$\nabla_2^2\psi = \left[\left(\frac{\mu}{m_1}\right)^2\nabla_R^2 + \frac{2\mu}{m_1}\nabla_R\cdot\nabla_r + \nabla_r^2\right]\psi$$

(#3) cont'd

Substituting into SE.

$$-\frac{\hbar^2}{2}\Big[\frac{\mu^2}{m_1 m_2^2}\nabla_R^2 + \frac{2\mu}{m_1 m_2}\nabla_R\cdot\nabla_r + \frac{1}{m_1}\nabla_r^2$$
$$+\frac{\mu^2}{m_1^2 m_2}\nabla_R^2 - \frac{2\mu}{m_1 m_2}\nabla_R\cdot\nabla_r + \frac{1}{m_2}\nabla_r^2\Big]$$
$$+V(r)\psi = E\psi$$

$$\Rightarrow -\frac{\hbar^2}{2}\Big[\frac{\mu^2}{m_1 m_2}\Big(\frac{1}{m_2}+\frac{1}{m_1}\Big)\nabla_R^2 + \Big(\frac{1}{m_1}+\frac{1}{m_2}\Big)\nabla_r^2\Big]\psi + V(r)\psi = E\psi$$

$\frac{m_1+m_2}{m_1 m_2}$ or $\frac{1}{\mu}$ $\qquad \mu = \frac{m_1 m_2}{m_1+m_2}$

$$\Rightarrow -\frac{\hbar^2}{2}\Big[\frac{\mu}{m_1 m_2}\nabla_R^2 + \frac{1}{\mu}\nabla_r^2\Big]\psi + V(r)\psi = E\psi$$

$$\boxed{-\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2\psi - \frac{\hbar^2}{2\mu}\nabla_r^2\psi + V(\bar{r})\psi = E\psi}$$

(2) Let $\psi(\bar{R},\bar{r}) = \psi_R(\bar{R})\,\psi_r(\bar{r})$

$$\underbrace{-\frac{\hbar^2}{2(m_1+m_2)}\frac{1}{\psi_R(R)}\nabla_R^2\psi_R(R)}_{=E_R} + \underbrace{\Big[-\frac{\hbar^2}{2\mu}\frac{1}{\psi_r(r)}\nabla_r^2\psi_r(r) + V(r)\Big]}_{=E_r} = E$$

depends only on $\bar{R}$ $\qquad$ depends only on $\bar{r}$

$$\boxed{-\frac{\hbar^2}{2\,m_{TOT}}\nabla_R^2\psi_R(R) = E_R\psi_R(R)}$$

$$\boxed{-\frac{\hbar^2}{2\mu}\nabla_r^2\psi_r(r) + V(r)\psi_r(r) = E_r\psi_r(r)}$$

#4 Griffiths Problem 5.2

Let m_e = electron mass m_p = proton mass

(a) $\mu = \dfrac{m_e m_p}{m_e + m_p}$ $E_1 = -m_e \alpha$ where $m = m_e + m_p$

$$\frac{\Delta E_1}{E} = \frac{m_e - \mu}{\mu} = \frac{m_e - \frac{m_e m_p}{m}}{\left(\frac{m_e m_p}{m}\right)} = \frac{m_e \overset{(m_e+m_p)}{m}}{m_e m_p} - \frac{m_e m_p}{m_e m_p}$$

$$= \frac{m_e^2}{m_e m_p} = \boxed{m_e/m_p} \approx \frac{9.11 \times 10^{-31}\,kg}{1.67 \times 10^{-27}\,kg} \approx \boxed{.0005}$$

(b) $n=3 \longrightarrow n=2$ Balmer lines

H: 1 proton : $\mu_H = \dfrac{m_e m_p}{m_e + m_p}$ D: Deuterium: 1 proton 1 neutron : $\mu_D = \dfrac{m_e 2m_p}{m_e + 2m_p}$

$$\frac{1}{\lambda_H} = R\left(\frac{1}{2^2} - \frac{1}{3^2}\right) \Longrightarrow \frac{1}{\lambda} = \left(\frac{5}{36}\right) R \Rightarrow \lambda_H = \frac{36}{5R}$$

$$1/\lambda = \alpha R^{-1} \Rightarrow d\lambda = -\alpha R^{-2} dR \text{ or } \frac{d\lambda}{\lambda} = -\frac{dR}{R}$$

$$d\lambda = -\lambda_H\left(\frac{dR}{R}\right). \quad \text{But } R \propto m \quad \frac{dR}{R} = \frac{\mu_D - \mu_H}{\mu_D}$$

$$\mu_D - \mu_H = \frac{2m_e m_p}{m_e + 2m_p} - \frac{m_e m_p}{m_e + m_p} = \frac{2m_e^2 m_p + 2m_e m_p^2 - m_e^2 m_p - 2m_e m_p^2}{(m_e + 2m_p)(m_e + m_p)}$$

$$= \frac{m_e^2 m_p}{(m_e + 2m_p)(m_e + m_p)} = \mu_H\left(\frac{m_e}{m_e + 2m_p}\right) \quad \frac{\mu_D - \mu_H}{\mu_H} = \left(\frac{m_e}{m_e + 2m_p}\right) \approx \frac{m_e}{2m_p}$$

$$d\lambda = -\lambda_H \frac{m_e}{2m_p} = -\left(\frac{36}{5R}\right)\frac{9.11 \times 10^{-31}\,kg}{2(1.67 \times 10^{-27}\,kg)} \approx \boxed{6.55 \times 10^{-7}\,m}$$

$R = 1.1 \times 10^7$

(8)

#4 continued

(c) Binding energy for H ~ 13.6eV

$$E_H = \cancel{\alpha}\, \mu_H = \alpha\left(\frac{m_e m_p}{m_e+m_p}\right) = 13.6eV$$

$$E_p = \alpha\, \mu_p = \alpha\left(\frac{m_e^2}{2m_e}\right) = \alpha\left(\frac{m_e}{2}\right)$$

$$\frac{E_p}{E_H} = \frac{\mu_p}{\mu_H} \Rightarrow E_p = 13.6eV\left(\frac{m_e}{2}\right)\left(\frac{m_e+m_p}{m_e m_p}\right)$$

$$E_p = 13.6eV\left(\frac{m_e+m_p}{2m_p}\right) \approx 13.6eV/2 = \boxed{6.8eV}$$

(d) From (b) $\frac{1}{\lambda_H} = R\left(1-\frac{1}{2^2}\right) = \frac{3}{4}R$ $\boxed{\lambda_H = \frac{4}{3R}}$

$$\mu_H = \frac{m_e m_p}{m_e+m_p} \qquad \mu_M = \frac{207 m_e m_p}{207m_e+m_p}$$

$\boxed{\begin{array}{l}\lambda_H = H \text{ wavelength} \\ \lambda_M = \text{Muonic } H \text{ wavelength}\end{array}}$

$$\lambda = \alpha R^{-1} \sim \beta \mu^{-1}$$

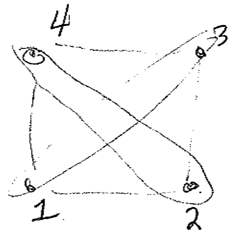
$$\frac{\lambda_M}{\lambda_H} = \frac{1/\mu_M}{1/\mu_H} = \mu_H/\mu_M \text{ or } \lambda_M = \lambda_H\left(\frac{\mu_H}{\mu_M}\right)$$

$$\lambda_M = \left(\frac{4}{3R}\right)\left(\frac{m_e m_p}{m_e+m_p}\right)\left(\frac{207m_e+m_p}{207 m_e m_p}\right) = \left(\frac{4}{3R}\right)\frac{207m_e+m_p}{207(m_e+m_p)}$$

$$\lambda_M = \frac{4}{3(1.01\times10^7)}\left[\frac{207(9.11\times10^{-31}kg)+1.67\times10^{-27}kg}{207(9.11+10^{-31}kg+1.67\times10^{-27}kg}\right]$$

$$\approx \boxed{6.5\times10^{-10}m}$$

#5 $\mathcal{H} = \frac{J}{\hbar^2}(S_1\cdot S_2 + S_2\cdot S_3 + S_3\cdot S_4 + S_4\cdot S_1) + \frac{J'}{\hbar^2}(S_1\cdot S_3 + S_2\cdot S_4)$



Now, $\bar{S}_i\cdot\bar{S}_j$ commutes with $\bar{S}_i + \bar{S}_j$

Recast $\mathcal{H}$ as

$$\mathcal{H} = \frac{J}{\hbar^2}(\overbrace{S_1 + S_3}^{S_{13}})\cdot(\overbrace{S_2 + S_4}^{S_{24}}) + \frac{J'}{\hbar^2}(S_1\cdot S_3 + S_2\cdot S_4)$$

Let $\left\{\begin{array}{l} S_{13} = S_1 + S_3 \text{ and } S_{24} = S_2 + S_4 \\ S_{TOT} = S_1 + S_2 + S_3 + S_4 \end{array}\right\}$ All Commute

(1) $S_{13}\cdot S_{24} = \frac{1}{2}\left[(\underbrace{S_{13}+S_{24}}_{S_{TOT}})^2 - S_{13}^2 - S_{24}^2\right]$

(2) $\bar{S}_1\cdot\bar{S}_3 = \frac{1}{2}\left[(\underbrace{S_1+S_3}_{S_{13}})^2 - S_1^2 - S_3^2\right]$

(3) $\bar{S}_2\cdot\bar{S}_4 = \frac{1}{2}\left[(\underbrace{S_2+S_4}_{S_{24}})^2 - S_2^2 - S_4^2\right]$

Now

$$\mathcal{H} = J\,\frac{1}{2}\left[\bar{S}_{TOT}^2 - (S_1+S_3)^2 - (S_2+S_4)^2\right] + \frac{J'}{2}\left[(S_1+S_3)^2 - S_1^2 - S_3^2 + (S_2+S_4)^2 - S_2^2 - S_4^2\right]$$

Let $\boxed{S_1 = S_2 = S_3 = S_4 \Rightarrow \text{spin } 1/2 \text{ operators}}$

$S_1 + S_3 = S_{13} \Rightarrow \frac{1}{2}\otimes\frac{1}{2} \Rightarrow S_{13} = 1, 0$

$S_2 + S_4 = S_{24} \Rightarrow \frac{1}{2}\otimes 1/2 \qquad S_{24} = 1, 0$

$S_{TOT} = S_{13} + S_{24} \Rightarrow \underset{(3+1)}{1\oplus 0}\otimes\underset{(3+1)}{1\oplus 0}$ → 16 states

$\Rightarrow S_{TOT} = 2, 1, 0$

Eigenvalues given by

$$\frac{J\hbar^2}{2}\left[S_{TOT}(S_{TOT}+1) - S_{13}(S_{13}+1) - S_{24}(S_{24}+1)\right] + \frac{J'\hbar}{2}\left[S_{13}(S_{13}+1) + S_{24}(S_{24}+1) - 4\overset{1/2}{S}(S+1)\right]$$

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Ground State: $J >> J' \approx 0$

$$\boxed{E = \frac{J\hbar^2}{2}\left[S_{TOT}(S_{TOT}+1) - S_{13}(S_{13}+1) - S_{24}(S_{24}+1)\right] + \frac{J'\hbar^2}{2}\left[S_{13}(S_{13}+1) + S_{24}(S_{24}+1) - 4\left(\frac{3}{4}\right)\right]}$$

Minimize S_{TOT} $\Rightarrow$ $S_{TOT} = 0$

Maximize S_{13}, S_{24} $\Rightarrow$ $S_{13}, S_{14} = 1$

$$E^1_{GS} = \frac{J\hbar^2}{2}\left[0 - 2 - 2\right] + \frac{J'\hbar^2}{2}\left[2 + 2 - 4\left(\frac{3}{4}\right)\right]$$

$$= -2J\hbar^2 + \frac{J'}{2}\hbar^2 = \hbar^2\left[-2J + \frac{J'}{2}\right]$$

Ground State: $J' >> J$

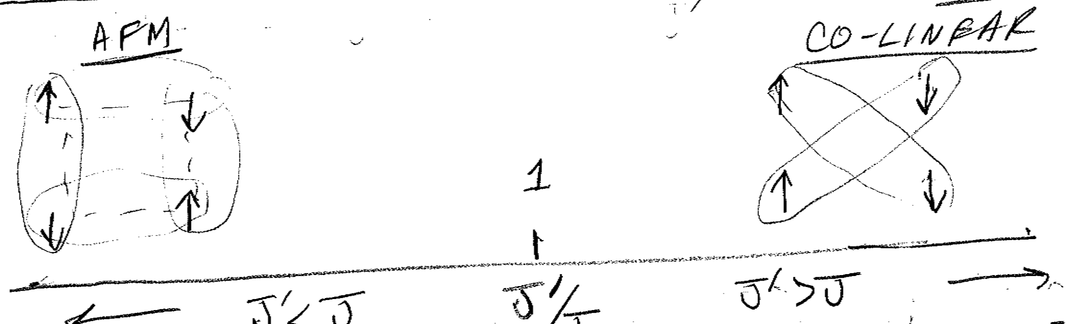
Minimize S_{13}, S_{24} $\Rightarrow$ $S_{13} = S_{24} = 0$ $\Rightarrow$ $S_{TOT} = 0$

(1,0) (1,0)

$$E^2_{GS} = \frac{J\hbar^2}{2}\left[0\right] + \frac{J'\hbar^2}{2}\left[0 + 0 - 3\right] = \frac{-3J'}{2}\hbar^2$$

Crossover occurs when $E^1_{GS} \longrightarrow E^2_{GS}$

or $-2J + J'/2 = -\frac{3J'}{2}$ or $J = J'$ $\quad (\alpha = 1)$



$S_{TOT} = 0$
$S_{13}, S_{24} = 1$

$S_{TOT} = 0$
$S_{13}, S_{24} = 0$