SOLUTIONS

**PROBLEM SET 6 Due Friday November 11**

Physics 115B– FALL 2011

November 11 is a holiday. I have therefore not included any numerical work in this assignment.

Points Assigned

Analytic:

10 **[1.]** Griffiths Problem 4.32

10 **[2.]** Griffiths Problem 4.35

10 **[3.]** Griffiths Problem 4.36

10 **[4.]** Griffiths Problem 4.49

10 **[5.]** In class we added two spin-1/2 and got spin-1 and spin-0 (very similar to Griffiths Sec. 4.4.3.) Follow the same process for adding spin-1 and spin-1/2. That is, write down expressions for eigenstates of the four (mutually commuting) operators $(S_1+S_2)^2, S_1^2, S_2^2, S_1^z+S_2^z$ in terms of the eigenstates of the four (mutually commuting) operators $S_1^2, S_2^2, S_1^z, S_2^z$.

50 Total

(#1) Griffiths 4.32:

(a)
$$\chi(t) = \begin{pmatrix} \cos(\alpha/2)\, e^{\frac{i\gamma B_0 t}{2}} \\ \sin(\alpha/2)\, e^{-\frac{i\gamma B_0 t}{2}} \end{pmatrix}, \qquad \chi_+^x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\text{Prob}\left(S_x = +\tfrac{\hbar}{2}\right) = \left|\langle \chi_+^x \mid \chi(t)\rangle\right|^2$$
$$= \frac{1}{2}\left[(1 \quad 1)\, \chi(t)\right]^2 = \frac{1}{2}\left[\cos\left(\tfrac{\alpha}{2}\right)e^{\frac{i\gamma B_0 t}{2}} + \sin\left(\tfrac{\alpha}{2}\right)e^{-\frac{i\gamma B_0 t}{2}}\right]^2$$

Let $\beta = \frac{\gamma B_0 t}{2}$ $\quad a = \cos\left(\frac{\alpha}{2}\right)$ $\quad b = \sin\left(\frac{\alpha}{2}\right)$

$$= \frac{1}{2}\left[a e^{-i\beta} + b e^{i\beta}\right]\left[a e^{i\beta} + b e^{-i\beta}\right]$$
$$= \frac{1}{2}\left[a^2 + b^2 + ab\, e^{-i2\beta} + ab\, e^{+i2\beta}\right]$$
$$= \frac{1}{2}\left[\cos^2\left(\tfrac{\alpha}{2}\right) + \sin^2\left(\tfrac{\alpha}{2}\right) + \underbrace{\cos\left(\tfrac{\alpha}{2}\right)\sin\left(\tfrac{\alpha}{2}\right)}_{(2\sin\alpha)}\tfrac{1}{2}\cos(\overset{2\beta}{\gamma B_0 t})\right]$$

$$\boxed{\text{Prob}\left(S_x = \tfrac{\hbar}{2}\right) = \frac{1}{2} + \frac{1}{2}\sin\alpha\cos(\gamma B_0 t)}$$

(b) $\chi_+^y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$

$$\text{Prob}\left(S_y = +\tfrac{\hbar}{2}\right) = \left|\langle \chi_+^y \mid \chi(t)\rangle\right|^2$$
$$= \frac{1}{2}\left[(1 \quad -i)\, \chi(t)\right] = \frac{1}{2}\left[\cos\left(\tfrac{\alpha}{2}\right)e^{\frac{i\gamma B_0 t}{2}} - i\sin\left(\tfrac{\alpha}{2}\right)e^{-\frac{i\gamma B_0 t}{2}}\right]$$
$$= \frac{1}{2}\left[a e^{-i\beta} + i b e^{+i\beta}\right]\left[a e^{i\beta} - i b e^{-i\beta}\right]$$
$$= \frac{1}{2}\left[\underbrace{a^2 + b^2}_{} - iab\, e^{-2i\beta} + iab\, e^{+2i\beta}\right]$$
$$= 1/2\left[\quad 1 \quad + ab\,\boxed{i\left[e^{+2i\beta} - e^{-2i\beta}\right]}\right] \leftarrow -\frac{\sin(2\beta)}{2}$$
$$= 1/2\left[\quad 1 + \underbrace{\cos\left(\tfrac{\alpha}{2}\right)\sin\left(\tfrac{\alpha}{2}\right)}_{2\sin(\alpha)} - \left(\frac{\sin(\gamma B_0 t)}{2}\right)\right]$$

$$\text{Prob}\left(S_y = +\hbar/2\right) = \boxed{\frac{1}{2} - \frac{1}{2}\sin\alpha\sin(\gamma B_0 t)}$$

## #1 Griffiths 4.32

$\text{Prob}(S_z = +\frac{\hbar}{2})$  $\chi_+^z = \begin{pmatrix}1\\0\end{pmatrix}$

$= |\langle \chi_+^z | \chi(t)\rangle|^2 = \langle (1\ 0)\ \chi(t)\rangle^2$

$= \left|\cos(\frac{\alpha}{2})\, e^{\frac{i\gamma B_0 t}{2}}\right|^2 = \boxed{\cos^2(\frac{\alpha}{2})}$

## #2 $\boxed{\text{Griffiths 4.35}}$

(a) Baryons - 3 quarks (spin 1/2) $\Rightarrow$ $\boxed{\text{Spins: } \frac{3}{2}, \frac{1}{2}}$

(b) Mesons - 2 quarks $\Longrightarrow$ $\boxed{\text{Spins: } 1, 0}$

## #3 $\boxed{\text{Griffiths 4.36}}$

(a) $S_1 = 1,\ S_2 = 2$  $S_1 + S_2 = 3$  $m_1 + m_2 = 1 \Rightarrow |3\ 1\rangle$

$$|3\ 1\rangle = \frac{1}{\sqrt{15}}\,|\overset{S_2}{2}\ \overset{S_1}{-1}\rangle + \sqrt{\frac{8}{15}}\,|1\ 0\rangle + \sqrt{\frac{6}{15}}\,|0\ 1\rangle$$

(From CG Table)

$$\boxed{\text{Prob } S_z^2 = \begin{cases} 2\hbar & 1/15 \\ 1\hbar & 8/15 \\ 0 & 6/15 \end{cases}}$$

(b) Electron $|\downarrow\rangle$  $\psi_{nlm} = \psi_{510}$  $l = 1,\ m = 0$

Find $\underset{1}{|1\,0\rangle}\ \underset{\times\ 1/2}{|\downarrow\rangle}$ in terms of total ang. momentum bases

$$|1\ 0\rangle|\downarrow\rangle = \sqrt{\frac{2}{3}}\,|3/2\ -1/2\rangle + \frac{1}{\sqrt{3}}\,|1/2\ -1/2\rangle$$

(From CG Table)

$$\boxed{\text{Prob } S_{TOT}^2 = \begin{cases} \hbar^2\frac{3}{2}(\frac{5}{2}) = \frac{15}{4}\hbar^2 \rightarrow 2/3 \\ \hbar^2\frac{1}{2}(\frac{3}{2}) = \frac{3}{4}\hbar^2 \rightarrow 1/3 \end{cases}}$$

#4 Griffiths 4.49

$$\chi = A \begin{pmatrix} 1-2i \\ 2 \end{pmatrix}$$

(a) $A^2 (1+2i \;\; 2) \begin{pmatrix} 1-2i \\ 2 \end{pmatrix} = A^2(5+4) = 9A^2 = 1$

$$\boxed{A = 1/3}$$

(b) Prob $S_z$ =

$\hbar/2 \quad = |\langle \chi_+^z | \chi \rangle|^2 = \left|(1 \; 0)\frac{1}{3}\begin{pmatrix} 1-2i \\ 2 \end{pmatrix}\right|^2 = \boxed{\frac{5}{9}}$

$-\hbar/2 \quad = \langle \chi_-^z | \chi \rangle|^2 = \left|(0 \; 1)\frac{1}{3}\begin{pmatrix} 1-2i \\ 2 \end{pmatrix}\right|^2 = \boxed{\frac{4}{9}}$

$$\langle S_z \rangle = \frac{5}{9}\left(\frac{\hbar}{2}\right) + \frac{4}{9}\left(-\frac{\hbar}{2}\right) = \boxed{\frac{\hbar}{18}}$$

(c) $\langle \chi_+^x | \chi \rangle = \frac{1}{3}\frac{1}{\sqrt{2}}(1 \; 1)\begin{pmatrix} 1-2i \\ 2 \end{pmatrix} = \frac{1}{3\sqrt{2}}(3-2i)$

$\text{Prob}(S_x = \hbar/2) = |\langle \chi_+^x | \chi \rangle|^2 = \frac{1}{18}(13) = \boxed{\frac{13}{18}} \rightarrow \hbar/2$

$\langle \chi_-^x | \chi \rangle = \frac{1}{3}\frac{1}{\sqrt{2}}(1 \; {-1})\begin{pmatrix} 1-2i \\ 2 \end{pmatrix} = \frac{1}{3\sqrt{2}}(-1-2i)$

$\text{Prob}(S_x = -\hbar/2) = |\langle \chi_-^x | \chi \rangle|^2 = \frac{1}{18}(5) = \boxed{\frac{5}{18}} \rightarrow -\frac{\hbar}{2}$

$\hookrightarrow \langle S_x \rangle = \frac{8\hbar}{36} = \boxed{\frac{2\hbar}{9}}$

(d) $\chi_+^y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \chi_-^y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$\langle \chi_+^y | \chi \rangle = \frac{1}{3}\frac{1}{\sqrt{2}}(1 \; {-i})\begin{pmatrix} 1-2i \\ 2 \end{pmatrix} = \frac{1}{3\sqrt{2}}(1-4i)$

$P(S^y = \hbar/2) = |\langle \chi_+^y | \chi \rangle|^2 = \frac{1}{18}(17) = \boxed{\frac{17}{18}} \rightarrow \hbar/2$

$\langle \chi_-^y | \chi \rangle = \frac{1}{3}\frac{1}{\sqrt{2}}(1 \; i)\begin{pmatrix} 1-2i \\ 2 \end{pmatrix} = \frac{1}{3\sqrt{2}}(1)$

$P(S^y = -\hbar/2) = |\langle \chi_-^y | \chi \rangle|^2 = \frac{1}{18}(1) \quad \boxed{\frac{1}{18}} \rightarrow -\hbar/2$

$$\langle S_y \rangle = \frac{16}{18}\left(\frac{\hbar}{2}\right) = \frac{8\hbar}{18} = \boxed{\frac{4}{9}\hbar}$$

#5 $S_1 = 1$, $S_2 = 1/2$

**Eigenstates of $(S_1+S_2)^2$**

$|3/2\ 3/2\rangle$ $\quad |\tfrac{1}{2}\ \tfrac{1}{2}\rangle$

$|3/2\ 1/2\rangle$

$|3/2\ -1/2\rangle$ $\quad |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$

$|3/2\ -3/2$

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**Eigenstates of $S_1^2, S_2^2, \ldots$**

$$\left.\begin{matrix}|1\,1\rangle\\ |1\,0\rangle\\ |1\,-1\rangle\end{matrix}\right\} \times \left[\begin{matrix}|\tfrac{1}{2},\tfrac{1}{2}\rangle\\ |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\end{matrix}\right] \Longrightarrow \begin{matrix}|1+\rangle & |1-\rangle\\ |0+\rangle & |0-\rangle\\ |-1+\rangle, & |-1-\rangle\end{matrix}$$

where $+ = \tfrac{1}{2}$, $- = -1/2$ 6

Start w/ $|3/2\ 3/2\rangle$ and apply $J^- = S_1^- + S_2^-$

$$\boxed{\begin{aligned} S^-|1\rangle &= \sqrt{2}\,|0\rangle\\ S^-|0\rangle &= \sqrt{2}\,|-1\rangle\\ S^-|+\rangle &= |-\rangle\end{aligned}}$$

$(S_1^- + S_2^-)$:

$$\boxed{|3/2\ 3/2\rangle = |1+\rangle}$$

$$|3/2\ 1/2\rangle = \sqrt{2}\,|0+\rangle + |1-\rangle$$

Normalize: $A^2 = 2+1 = 3$

$$\boxed{|3/2\ 1/2\rangle = \sqrt{\tfrac{2}{3}}\,|0+\rangle + \tfrac{1}{\sqrt{3}}\,|1-\rangle}$$

$$|3/2\ -1/2\rangle = \sqrt{\tfrac{2}{3}}\left(\sqrt{2}\,|-1+\rangle + |0-\rangle\right) + \tfrac{1}{\sqrt{3}}\left(\sqrt{2}\,|0-\rangle\right)$$

$$= \tfrac{2}{\sqrt{3}}\,|-1+\rangle + \tfrac{2\sqrt{2}}{\sqrt{3}}\,|0-\rangle$$

Normalize: $A^2 = \tfrac{4}{3} + \tfrac{8}{3} = 4$

Devide by $\tfrac{1}{A}$!

$$\boxed{|3/2\ -1/2\rangle = \tfrac{1}{\sqrt{3}}\,|-1+\rangle + \sqrt{\tfrac{2}{3}}\,|0-\rangle}$$

$$|3/2\ -3/2\rangle = \tfrac{1}{\sqrt{3}}\,|-1-\rangle + \sqrt{\tfrac{2}{3}}\,\sqrt{2}\,|-1-\rangle$$

$\to$ Normalize: only 1 state $A = 1$

$$\boxed{|3/2\ -3/2\rangle = |-1-\rangle}$$

State orthogonal to $|3/2\ 1/2\rangle$

$$\boxed{|1/2\ 1/2\rangle = -\sqrt{\tfrac{1}{3}}\,|0+\rangle + \sqrt{\tfrac{2}{3}}\,|1-\rangle}$$

$$|1/2\ -1/2\rangle = -\sqrt{\tfrac{1}{3}}\left(\sqrt{2}\,|-1+\rangle + |0-\rangle\right) + \sqrt{\tfrac{2}{3}}\left(\sqrt{2}\,|0-\rangle\right)$$

$$\boxed{|1/2\ -1/2\rangle = -\sqrt{\tfrac{2}{3}}\,|-1+\rangle + \tfrac{1}{\sqrt{3}}\,|0-\rangle}$$

All values check out w/ Clebsch-Gordan Tables!