$\boxed{\text{How to include } \vec{E}, \vec{B} \text{ fields in QM?}}$

$\vec{E}$ is sort of obvious: $\vec{E} \rightarrow -\vec{\nabla} V(\vec{r})$

so put this $V(\vec{r})$ into Sch. Eqn as usual (actually $-eV(r)$)

What about $\vec{B}$? Answer is

$$\vec{p} \rightarrow \vec{p} - e/c\,\vec{A}$$

where $\vec{B} = \vec{\nabla} \times \vec{A}$. We will discuss in a

bit why this might make sense, but first note

issue of gauge tranformations $\vec{A} \rightarrow \vec{A} + \vec{\nabla}\Lambda$

yields same $\vec{B}$ since $\vec{\nabla} \times \vec{\nabla} = 0$. Thus appears

to be ambiguity or at least one needs to show

Sch Eqn gives same eigenfunctions, eigenvalues under

gauge transformation.

B-2

Consider $\vec{B} = B_0 \hat{z}$

one possible $\vec{A} = -\frac{1}{2} B_0 y \hat{x} + \frac{1}{2} B_0 x \hat{y}$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{1}{2} B_0 y & \frac{1}{2} B_0 x & 0 \end{vmatrix} = B_0 \hat{z}$$

Then $\frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 = \frac{1}{2}\left[\left(p_x + \frac{eB_0}{2mc} y\right)^2 + \left(p_y - \frac{eB_0}{2mc} x\right)^2 + p_z^2\right]$

since $[p_x, y] = [p_y, x] = 0$ we get

$$\frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + \frac{eB_0}{2mc}\left(y p_x - x p_y\right)$$
$$+ \frac{e^2 B_0^2}{8mc^2}\left(x^2 + y^2\right)$$

$$= \frac{1}{2m}\vec{p}^{\,2} + \frac{eB_0}{2mc} L_z + \frac{e^2 B_0^2}{8mc^2}(x^2 + y^2)$$

$\vec{B} = 0$ form

Why might this make sense? If $\vec{B} = B_0 \hat{z}$

classically e^- will have circular orbit in xy plane

(if we set $v_z = 0$). The $B_0^2 e^2 / 8mc^2 \, (x^2 + y^2)$ is a

confining (harmonic oscillator potential which gives "circular orbits"

B-3

$[a, a^\dagger] = 1$

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a_x + a_x^\dagger\right)$$

<u>Check</u> $[x, p_x]$

$$p_x = i\sqrt{\frac{\hbar m\omega}{2}}\left(a_x^\dagger - a_x\right)$$

$$= i\frac{\hbar}{2}\left(1-(-1)\right) = \hbar i \quad \checkmark$$

$$Y p_x - X p_y = i\frac{\hbar}{2}\left((a_y + a_y^\dagger)(a_x^\dagger - a_x) - (a_x + a_x^\dagger)(a_y^\dagger - a_y)\right)$$

$$= i\frac{\hbar}{2}\Big(a_y a_x^\dagger + a_y^\dagger a_x^\dagger - a_y a_x - a_y^\dagger a_x$$

$$+ a_x^\dagger a_y - a_x^\dagger a_y^\dagger + a_x a_y - a_x a_y^\dagger \Big)$$

$$= i\hbar\left(a_x^\dagger a_y - a_y^\dagger a_x\right)$$

$$H = \frac{p_z^2}{2m} + p_x^2 + \frac{1}{2}m\omega^2 x^2 + p_y^2 + \frac{1}{2}m\omega^2 y^2 + \frac{eB_0}{2mc}L_z$$

$\hbar\omega(a_x^\dagger a_x + 1/2)$ $\quad$ $\hbar\omega(a_y^\dagger a_y + \frac{1}{2})$

$$\frac{1}{2}m\omega^2 \equiv \frac{e^2 B_0^2}{8mc^2} \qquad \omega^2 = \frac{e^2 B_0^2}{4m^2c^2} \qquad \omega = \frac{eB_0}{2mc}$$

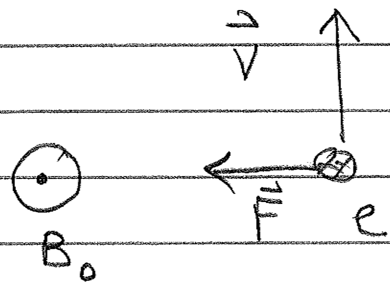
"cyclotron frequency"

$$\omega_c = \frac{eB_0}{mc}$$

See B-3A

B3A

Classical analysis



$$\vec{F} = -e\vec{v} \times \vec{B}_0 \frac{1}{c}$$

Circular orbit

$$\frac{mv^2}{r} = \frac{evB_0}{c}$$

$$m\omega = \frac{eB_0}{c}$$

cyclotron frequency $\quad \omega_c = \frac{eB_0}{mc}$

$$a_r = (a_x + i a_y)\frac{1}{\sqrt{2}}$$

$$a_\ell = (a_x - i a_y)\frac{1}{\sqrt{2}}$$

$$a_r^\dagger a_r + a_\ell^\dagger a_\ell = \frac{1}{2}\left[(a_x^\dagger - i a_y^\dagger)(a_x + i a_y) + (a_x^\dagger + i a_y^\dagger)(a_x - i a_y)\right]$$

$$= \frac{1}{2}\left[a_x^\dagger a_x - i a_y^\dagger a_x + i a_x^\dagger a_y + a_y^\dagger a_y\right.$$

$$\left. + a_x^\dagger a_x + i a_y^\dagger a_x - i a_x^\dagger a_y + a_y^\dagger a_y\right] = a_x^\dagger a_x + a_y^\dagger a_y$$

Also $\quad a_r^\dagger a_r - a_\ell^\dagger a_\ell = i\left(a_x^\dagger a_y - a_y^\dagger a_x\right)$

So $\quad \hat{H} = \dfrac{P_z^2}{2m} + \hbar\omega\left(a_r^\dagger a_r + \frac{1}{2}\right) + \hbar\omega\left(a_\ell^\dagger a_\ell + \frac{1}{2}\right)$

$$+ \hbar\omega\left(a_r^\dagger a_r - a_\ell^\dagger a_\ell\right)$$

$$\hat{H} = \frac{\hat{P}_z^2}{2m} + \underbrace{2\hbar\omega}_{\hbar\frac{eB_0}{mc}}\, a_r^\dagger a_r + \frac{1}{2}\hbar\omega$$

$$+ \hbar\omega_c\, a_r^\dagger a_r$$

funny factor of 2 goes away!

B-5

$$\hat{H} = \frac{\hat{p}_z^2}{2m} + \hbar\omega_c a_r^\dagger a_r + \frac{1}{4}\hbar\omega_c$$

No dependence on $a_\ell, a_\ell^\dagger$ operators!!!

Eigenstates

$$|\psi\rangle_{k_z n_\ell n_r} = e^{ik_z z}\,|n_\ell\rangle|n_r\rangle$$

$$|n_\ell\rangle \sim (a_\ell^\dagger)^{n_\ell}\,|VAC\rangle$$

$$|n_r\rangle \sim (a_r^\dagger)^{n_r}\,|VAC\rangle$$

$$E_{k_z n_\ell n_r} = \frac{\hbar^2 k_z^2}{2m} + \hbar\omega_c n_r + \frac{1}{2}\hbar\omega_c$$

∞ degeneracy (any n_ℓ gives same E!)

Physically: cyclotron orbit can be anywhere in xy plane