project: AM - 1

SHO: solve Sch. Eqn, a pde
— or — "algebraic method"
introduce $a, a^\dagger$ and write $\hat{H}$ in terms ~~of them and use [ ] relns~~
SAME HERE. ~~start with pde approach~~

Angular Momentum

we found $\Psi_{n\ell m}(r\theta\phi) = R_{n\ell}(r) Y_{\ell m}(\theta,\phi)$

by separation of variables considering only $\hat{H}$ in spherical coordinates. Now it's time to see how $\vec{L}$ enters.

$$\vec{L} = \vec{r} \times \vec{p}$$

$$L_z = x p_y - y p_x = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \qquad \text{ok } L_x = \ldots,\ L_y = \ldots$$

Many ways to do algebra. I will do something a bit different from Griffiths 4.3.2 but look at that too

$$x = r\sin\theta\cos\phi \qquad r^2 = x^2 + y^2 + z^2$$

$$y = r\sin\theta\sin\phi \qquad \tan\phi = y/x$$

$$z = r\cos\theta \qquad \tan\theta = \left(\frac{x^2+y^2}{z^2}\right)^{1/2}$$

Recall $\frac{d}{du}\tan^{-1}u = \frac{1}{1+u^2}$

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$$\frac{\partial}{\partial y} = \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} + \frac{\partial r}{\partial y}\frac{\partial}{\partial r}$$

$$= \frac{1}{1+\frac{x^2+y^2}{z^2}}\,\frac{1}{2}\left(\frac{x^2+y^2}{z^2}\right)^{-1/2}\frac{2y}{z^2}\frac{\partial}{\partial \theta} + \frac{1}{1+\frac{y^2}{x^2}}\,\frac{1}{x}\frac{\partial}{\partial \phi} + \frac{y}{r}\frac{\partial}{\partial r}$$

$$= \frac{z}{\sqrt{x^2+y^2}}\,\frac{y}{r^2}\frac{\partial}{\partial \theta} + \frac{x}{x^2+y^2}\frac{\partial}{\partial \phi} + \frac{y}{r}\frac{\partial}{\partial r}$$

Similarly

$$\frac{\partial}{\partial x} = \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} + \frac{\partial r}{\partial x}\frac{\partial}{\partial r}$$

$$= \frac{1}{1+\frac{x^2+y^2}{z^2}}\,\frac{1}{2}\left(\frac{x^2+y^2}{z^2}\right)^{-1/2}\frac{2x}{z^2}\frac{\partial}{\partial \theta} + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right)\frac{\partial}{\partial \phi} + \frac{x}{r}\frac{\partial}{\partial r}$$

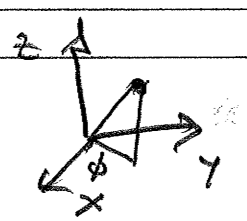
$$= \frac{z}{\sqrt{x^2+y^2}}\,\frac{x}{r^2}\frac{\partial}{\partial \theta} - \frac{y}{x^2+y^2}\frac{\partial}{\partial \phi} + \frac{x}{r}\frac{\partial}{\partial r}$$

Now watch! $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ ← $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial r}$ terms cancel!

$$= \frac{\partial}{\partial \phi} \quad !$$

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial \phi}$$ very simple

Reason is ϕ is rotation about $\hat{z}$ axis of course!



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project:

AM-3

Unsurprisingly L_x and L_y more complex!

But notice already something interesting

$$Y_{\ell m}(\theta,\phi) = P_\ell^m(\cos\theta)\, e^{im\phi}$$

$$L_z Y_{\ell m}(\theta,\phi) = m\hbar \; \underbrace{P_\ell^m(\cos\theta)\, e^{im\phi}}_{\therefore \text{ eigenfunction}}$$

eigenvalue.

The eigenvalues of L_z are $m\hbar$ where $m = -\ell, -\ell+1, \ldots +\ell$

ℓ = integer

In a few lectures we will attack angular momentum by "algebraic" approach analogous to raising/lowering operators in SHO. We will learn that ℓ can be half integer also "SPIN"!!

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project:

AM - 4

It is straightforward to do same algebra for L_x and L_y. No harder, but just a lot less cancellation at end of day

$$L_x = \frac{\hbar}{i}\left(-\sin\phi \frac{\partial}{\partial\theta} - \cos\phi \cot\theta \frac{\partial}{\partial\phi}\right)$$

$$L_y = \frac{\hbar}{i}\left(\cos\phi \frac{\partial}{\partial\theta} - \sin\phi \cot\theta \frac{\partial}{\partial\phi}\right)$$

[Note : Griffiths arrives at this a different way, but involves remembering $\vec{\nabla}$ in spherical coordinates. I prefer this since I remember $(r,\theta,\phi) \leftrightarrow (x\,y\,z)$ but not $\vec{\nabla}$ formula]

The final step is constructing $L_x^2 + L_y^2 + L_z^2$

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We found eigenstates of L_z.

Are they eigenstates of L_x and L_y also?

No! L_x and L_y do not commute with L_z so they cannot have same complete set of eigenfunctions WHY?

Can non-commuting operators/matrices share a few eigenfunctions/vectors?

$$[L_x, L_y] = [yp_z - zp_y, zp_x - xp_z]$$

$$= [yp_z, zp_x] - [zp_y, zp_x] - [yp_z, xp_z] + [zp_y, xp_z]$$

Only $[\,,\,] \neq 0$ are $[x, p_x]$ $[y, p_y]$ $[z, p_z]$

$$= yp_x[p_z, z] + xp_y[z, p_z]$$

$$= i\hbar(xp_y - yp_x) = i\hbar L_z \quad !$$

Must be true that $[L_y, L_z] = i\hbar L_x$

$[L_z, L_x] = i\hbar L_y$ } cyclic permutations

CANNOT KNOW L_x, L_y, L_z simultaneously!

project:

AM-5

What about $L^2 = L_x^2 + L_y^2 + L_z^2$

$$[L^2, L_x] = \underbrace{[L_x^2, L_x]}_{0} + [L_y^2, L_x] + [L_z^2, L_x]$$

$$[AB, C] = ABC - CAB$$

$$= ABC \underbrace{- ACB + ACB}_{} - CAB$$

$$= A[B, C] + [A, C]B$$

Aside When you describe fermionic particles (electrons, quarks, ...) in QFT you will encounter anticommutation relns $\{A, B\} \equiv$? guess

$$\{A, B\} = AB + BA$$

$[AB, C] = ABC - CAB = ABC + ACB - ACB - CAB = A\{B, C\} - \{A, C\}B$

Why? In sho $a^\dagger |0\rangle = |1\rangle$ $a^\dagger a^\dagger |0\rangle = a^\dagger |1\rangle = \sqrt{2}\,|2\rangle$

Fermions, → PAULI → $|2\rangle$ cannot be!

i.e. $c^\dagger c^\dagger = 0$ $\{c^\dagger, c^\dagger\} = 2c^\dagger c^\dagger = 0$

$$\{c_p^\dagger, c_k^\dagger\} = \delta_{pk} \quad \text{etc}$$

Much of QFT follows from !!

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project:

AM-6

Back to $[L^2, L_x] = [L_y^2, L_y] + [L_z^2, L_x]$

$$= L_y [L_y, L_x] + [L_y, L_x] L_y$$

$$+ L_z [L_z, L_x] + [L_z, L_x] L_z$$

$$= L_y(-i\hbar L_z) + (-i\hbar L_z) L_y$$

$$+ L_z i\hbar L_y + i\hbar L_y L_z = \emptyset$$

Of course $[L^2, L_y] = [L^2, L_z]$

So maybe we should search for eigenstates common to L_z and L^2. (Best we can do)

Let me state the answer. Can anyone guess?

Simultaneous eigenstates of L_z and L^2 are $Y_{\ell m}(\theta, \phi)$!!

$$L_z Y_{\ell m}(\theta,\phi) = m\hbar Y_{\ell m}(\theta,\phi)$$

$$L^2 Y_{\ell m}(\theta,\phi) = \ell(\ell+1)\hbar^2 Y_{\ell m}(\theta,\phi)$$

Easy to verify this

$$Y_{\ell m}(\theta,\phi) = P_\ell^m(\theta) e^{im\phi} \qquad L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

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project:

AM-7

To verify $L^2 Y_{\ell m}(\theta,\phi) = \ell(\ell+1)\hbar^2 Y_{\ell m}(\theta,\phi)$

involves writing L^2 as $\partial/\partial\theta$, $\partial/\partial\phi$ etc.

We can do it, but it is messy...

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

Compare to ∇^2

$$\nabla^2 = \underbrace{\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2}\frac{\partial^2}{\partial\phi^2}\right]}_{\text{BASICALLY THE SAME...}} + \underbrace{\frac{1}{r^2}\left(\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\right)}_{\text{goes away after separation of variables}}$$

$\therefore$ will not pursue this because we already did all this algebra a few weeks ago.

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