Assignment Five, Due Friday, February 16, 5:00 pm.

[1.] Griffiths Problem 7, Chapter 3.

[2.] Griffiths Problem 12, Chapter 3.

[3.] Find the analytic solution of the one-dimensional Poisson equation with $\rho(x) = 12 \epsilon_0 x^2$,

$$-\frac{d^2 \phi}{dx^2} = 12 x^2,$$

with boundary conditions $\phi(0) = \phi(1) = 0$. What is the value of the quantity,

$$E = \int_0^1 [\frac{1}{2} \left(\frac{d\phi}{dx}\right)^2 - 12 x^2 \phi ]dx$$

and what is its physical significance?

[4.] (extra credit) Solve problem 3 numerically. Use $dx = 0.1$. Make a plot containing $\phi(x)$ at a few times during the course of the iteration, and also containing the analytic solution. (I will talk about this problem on Wednesday in the “problem session” if you need guidance.)

[5.] (extra, extra credit) Solve Laplace’s equation $\nabla^2 \phi = 0$ for the potential $\phi(x, y)$ for the square region in the figure, with the boundary conditions shown. Use the iterative method discussed in class, whose form in $d = 2$ (with $\rho = 0$) is,

$$\phi(i, j) = \frac{1}{4}[\phi(i + 1, j) + \phi(i - 1, j) + \phi(i, j - 1) + \phi(i, j + 1)].$$

Use $dx = 0.1$ and $a = 1$. Compare your solution with the analytic one obtained in class by plotting the two results for $\phi(x, y = a/2)$, $0 < x < a$ on the same graph. Make a second graph comparing the analytic and numeric solutions for a vertical cut, $\phi(x = a/2, y)$, $0 < y < a$. 

![Diagram of a square region with potential boundaries](image)
1. **Griffiths 3-7**

Clearly there must be image charges below the plane as shown here: Then the force on $+q$ is

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \sum \frac{q^2}{(2d)^2} + \frac{q^2}{(4d)^2} - \frac{1}{(6d)^2}$$

$$= \frac{q^2}{4\pi\epsilon_0 d^2} \left\{ \frac{1}{2} + \frac{1}{8} - \frac{1}{36} \right\}$$

$$\mathbf{E} = \frac{1}{72} \left\{ -36 + \frac{9 - 2}{3} \right\}$$

$$(-\frac{29}{32})$$

2. **Griffiths 3-12**

As mentioned in my email, the first step here is to do problem 2.52. Let's do it! We are asked for the potential of two long wires running along $x$ axis.

We know from Gauss' law

$$E_{2\xi} = \frac{\Delta L}{\epsilon_0} \Rightarrow E = \frac{1}{2\pi\epsilon_0 r}$$

is field of long straight wire.
Hence the potential is

\[ V(r) = - \int E(r) dr = - \frac{d}{2\pi \varepsilon_0} \ln \frac{r}{r_0} \]

some origin

For a pair of wires with \( \pm \lambda \)

\[ \text{out } P : \quad V(r) = - \frac{\lambda}{2\pi \varepsilon_0} \left( \ln \frac{r_+}{r_0} - \ln \frac{r_-}{r_0} \right) \]

\[ = \frac{\lambda}{2\pi \varepsilon_0} \ln \left( \frac{r_-}{r_+} \right) \quad \text{origin } r_0 \text{ goes away!} \]

\[ V(y, z) = \frac{d}{4\pi \varepsilon_0} \ln \frac{(y+a)^2 + z^2}{(y-a)^2 + z^2} \]

If \( V = \text{const} = V_0 \) we need

\[ \frac{(y+a)^2 + z^2}{(y-a)^2 + z^2} = \frac{4\pi \varepsilon_0 V_0 / \lambda}{b} \]

\[ (y+a)^2 + z^2 = b(y-a)^2 + z^2 \]

\( b > 1 \) if \( V_0 > 0 \)

\( b < 1 \) if \( V_0 < 0 \)

\[ y^2 + 2ay + a^2 + z^2 = by^2 - 2ab y + ba^2 + b^2 \]

\[ (1-b) y^2 + da(1+b)y + (1-b)2^2 + (1-b)a^2 = 0 \]
Complete the square

\[
(1-b)\left[y + \frac{a(1+b)}{1-b}\right]^2 - \frac{a^2(1+b)^2}{(1-b)}
\]

\[
+ (1-b)x^2 + (1-b)y^2 = 0
\]

\[
\left[y + \frac{a(1+b)}{1-b}\right]^2 + x^2 = \frac{1}{1-b}\left[\frac{a^2(1+b)^2}{1-b} - (1-b)y^2\right]
\]

\Rightarrow \text{circle with center at } (y_0, z_0)

\[
y_0 = a \frac{(1+b)}{1-b}
\]

\[
z_0 = 0
\]

\[
y_0 = a \frac{(1+b)^2}{(1-b)^2}
\]

\[
z_0 = a^2 \frac{4b}{(1-b)^2}
\]

The radius is square root of rhs

\[
R^2 = \frac{a^2}{(1-b)^2} \left\{ (1+b)^2 - (1-b)^2 \right\} = \frac{4a^2b}{(1-b)^2}
\]

\[
R = 2a \sqrt{b} / (1-b)
\]

These eqns tell us equipotentials are cylinders with

axes along \( \hat{z} \), but notice center \( y_0 \neq a \)!

The cylinders are not centered on the lines of change.

(This is a bit counterintuitive at first, but not if

you think carefully!)
We now find 3-12 Easy.

\[ y_0 = \frac{a(1+b)}{1-b} \quad R = \frac{2a\sqrt{b}}{1-b} \quad b = e^{4\pi\varepsilon_0 V_0/\lambda} \]

Eqn for position and radius of equipotential in terms of position \( a \) of lines and potential \( V_0 \).

The location of the center of the equipotential is called \( d \) in 3-11 and is \( y_0 \) in 2-52.

\[ a(1+b)/1-b = d \]

In problem 2-52 we saw \( a^2 + y_0^2 = R^2 \).

So we immediately get \( a^2 = R^2 - d^2 \quad a = \sqrt{R^2 - d^2} \)

The other condition is

\[ \frac{y_0 - d}{R} = \frac{a(1+b)/(1-b)}{2a\sqrt{b}/(1-b)} = \frac{1}{2} \frac{1+b}{\sqrt{b}} = \frac{1}{2}(\sqrt{b} + \frac{1}{\sqrt{b}}) \]

But \( b = e^{4\pi\varepsilon_0 V_0/\lambda} \) so \( \sqrt{b} + \frac{1}{\sqrt{b}} = 2 \cosh \frac{2\pi\varepsilon_0 V_0}{\lambda} \)

So given \( d, R, V_0 \) we get \( \lambda \) from

\[ d/R = 2 \cosh \left( \frac{2\pi\varepsilon_0 V_0}{\lambda} \right) \quad \lambda = \frac{2\pi\varepsilon_0 V_0}{\cosh^{-1}(d/R)} \]
\[ -d^2 \frac{\phi}{dx^2} = +12x^2 \]

\[ \frac{d\phi}{dx} = A - 4x^3 \]

\[ \phi = Ax - x^4 + B \]

\[ \phi(0) = B = 0 \]

\[ \phi(1) = A - 1 = 0 \quad A = 1 \]

\[ \phi(x) = x - x^4 \]

\[ E = \int_0^1 \left[ \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 - 12x^2 \right] dx \]

\[ = \int_0^1 \left[ \frac{1}{2} \left( 1 - 4x^3 \right)^2 - 12x^2 \left( x - x^4 \right) \right] dx \]

\[ = \int_0^1 \left( \frac{1}{2} - 4x^3 + 8x^2 - 12x^3 + 12x^4 \right) dx \]

\[ = \left. \int_0^1 \left( \frac{1}{2} - 16x^3 + 12x^4 \right) dx \right| \]

\[ = \left. \frac{1}{2}x - 4x^4 + \frac{2}{7}x^7 \right|_0^1 \]

\[ = \frac{1}{2} - 4 + \frac{2}{7} = \frac{1}{14} \left\{ 7 - 56 + 40 \right\} = -\frac{9}{14}. \]

This looks just like a classical mechanics problem (calculus of variations) where we are asked to minimize

\[ S = \int \left( \frac{1}{2} (\frac{dx}{dt})^2 - U(x) \right) dx = \int \mathcal{L}(x) dx \]

The Lagrange eqn is

\[ \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) - \frac{\partial \mathcal{L}}{\partial x} = 0 \]

In our case, translating notation:

\[ \frac{d}{dx} \frac{d\phi}{dx} - (-12x^2) = 0 \]

\[ x \rightarrow \phi \quad t \rightarrow x \]

So: physical interpretation: Solve b Laplace

\[ d^2\phi/dx^2 = -12x^2 \]

\( \phi \) minimizes \( E \).
implicit none
integer it, Nit, i, N
real*8 h, phi(0:1000), s(0:1000), onemw, h2, w
real*8 phinew(0:1000), E, wo2, x

write (6,*) 'enter N,h'
read (5,*) N, h
write (67,*) 'N,h'
write (67,*) N, h
write (6,*) 'enter Nit,w'
read (5,*) Nit, w
write (67,*) 'Nit,w'
write (67,*) Nit, w

do 100 i=0, N
   phi(i)=0.d0
   phinew(i)=0.d0
   x=dxfloat(i)*h
   s(i)=12.d0*x*x
100 continue
E=0.d0
onemw=1.d0-w
wo2=w/2.d0
h2=h*h

do 1000 it=1, Nit
   phinew(0)=onemw*phi(0)+wo2*(phi(1)+h2*s(0))
do 200 i=1, N-1
      phinew(i)=onemw*phi(i)+wo2*(phi(i-1)+phi(i+1))
      +h2*s(i)
200 continue
phinew(N)=onemw*phi(N)+wo2*(phi(N-1)+h2*s(N))

do 300 i=1, N-1
   phi(i)=phinew(i)
300 continue
   if (N,le.10) write (68,888) (phi(i),i=0,N)
888 format(11f7.4)
E=0.d0
   do 400 i=1, N-1
      E=E+0.5d0*(phi(i+1)-phi(i))*(phi(i+1)-phi(i))
      -h2*s(i)*phi(i)
400 continue
write (67,990) it, E/h
write (6,990) it, E/h
990 format(i8,f12.8)
1000 continue

write (67,*) '
   do 1100 i=0, N
      write (67,991) i, dxfloat(i)*h, phi(i)
1100 continue
991 format(i8,2f12.8)
992 format(2f12.8)

h=0.005
   do 1200 i=0, 200
      x=dxfloat(i)*h
      write (67,992) x, x*x*x*x*x
1200 continue
continue
end
$d^2\phi/dx^2 = -12x^2$