[1.] Compute \( \ln(24 - 7i) \).

[2.] (a) Compute the eigenvalues and eigenvectors of the matrix,

\[
M = \begin{pmatrix}
8 & 12 & 0 \\
12 & 1 & 0 \\
0 & 0 & 7
\end{pmatrix}
\]

What relation do the eigenvectors have relative to each other? Does the form of \( M \) have anything to do with your answer?

[3.] Consider the force

\[
\vec{F} = 4y \hat{i} - 2x \hat{j}
\]

Compute the integral of \( \vec{F} \) from \((0, 0)\) to \((1, 2)\)
(a) along the path \( y = 2x \); and
(b) along the path \( y = 2x^2 \).
Describe any relevant general theorem which provides insight into your results.

[4.] Compute

\[
\int_{0}^{2\pi} \frac{d\theta}{5 + 4 \cos \theta}
\]

[5.] (a) State De-Moivre’s theorem.
(b) What is the basic idea behind proving it?
(c) Use De-Moivre’s theorem to prove \( \cos(2\theta) = \cos^2 \theta - \sin^2 \theta \).

[6.] You were listening to the election news on Wednesday, and the television announcer asserted that the entire population of Florida had promised to vote completely randomly, with probability \( p = 1/2 \) for Democrat Andrew Gillum and \( q = 1/2 \) for Republican Ron DeSantis. The final vote was 2,100,000 to 1,900,000. How likely is it the people of Florida kept their promise? Explain. (I expecting just 2-3 sentences here which enunciate an important general principle and then show how it applies to this situation.)
Physics 104A
Midterm Exam Solutions

1. \(24 - 7i = re^{i\theta}\) where

\[
r = \sqrt{24^2 + (-7)^2} = 25
\]

\[
\theta = \tan^{-1}\left(-\frac{7}{24}\right) = 0.284 \text{ radians}
\]

\[
\ln(re^{i\theta}) = \ln r + i\theta
\]

so \(\ln(24 - 7i) = \ln 25 - i0.284\)

= \(3.219 - i0.284\)

Could check by exponentiating eg

\[
e^{3.219} = 25
\]

\[
e^{-i0.284} = \cos(0.284) - i\sin(0.284)
\]

= \(0.96 - i0.28\)

\[
e^{3.219 - i0.284} = 25(0.96 - i0.28)
\]

= \(24 - 7i\) \(\checkmark\)
Eigenvalues are determined by 
\[ |M - \lambda I| = 0 \]
\[
\begin{vmatrix}
8 - \lambda & 12 & 0 \\
12 & 1 - \lambda & 0 \\
0 & 0 & 7 - \lambda \\
\end{vmatrix} = 0
\]

\[
(7 - \lambda) [(8 - \lambda)(1 - \lambda) - 144] = 0
\]

\[
(7 - \lambda) [\lambda^2 - 9\lambda - 136] = 0
\]

\[
(\lambda - 17)(\lambda + 8)
\]

Eigenvalues \( \lambda = 7, 17, -8 \)

For \( \lambda = 7 \) eigenvector \( \mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \) must obey

\[
\begin{pmatrix}
1 & 12 & 0 \\
12 & -6 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

This eqn says nothing about \( c \) but gives \( a = b = 0 \) because we have two homogeneous eqns in two unknowns which are linearly independent:

\[ a + 12b = 0 \implies a = b = 0 \]

\[ 12a - 6b = 0 \]

So the eigenvector for \( \lambda = 7 \) is \( \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \) normalized to length 1
For $\lambda = 17$:
\[
\begin{pmatrix}
-9 & 12 & 0 \\
12 & -16 & 0 \\
0 & 0 & -10
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

We immediately see $c = 0$. The egs $a$ and $b$ are dependent.

3 \((-3a + 4b) = 0 \quad$ first eqn

-4 \((-3a + 4b) = 0 \quad$ second eqn

So we have $-3a + 4b = 0 \Rightarrow \frac{1}{5} \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} = \vec{v}_2$

is (normalized) eigenvector.

For $\lambda = -8$:
\[
\begin{pmatrix}
16 & 12 & 0 \\
12 & 9 & 0 \\
0 & 0 & 15
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

Again, $c = 0$ and dependent eqns for $a, b$

$4a + 3b = 0 \Rightarrow \frac{1}{5} \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix} = \vec{v}_3$

is eigenvector.

The eigenvectors are perpendicular to each other. Eq.
\[
\vec{v}_1 \cdot \vec{v}_2 = (0, 0, 1) \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} = 0
\]
\[
\vec{v}_2 \cdot \vec{v}_3 = (4, 3, 0) \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix} = 0
\]

This must occur because $M$ is Hermitian (real symmetric).
\[ \overrightarrow{F} = 4y \hat{z} - 2x \hat{j} \]

\[ a) \quad y = 2x \]
\[ dy = 2dx \]
\[ \int \overrightarrow{F} \cdot dr = \int F_x \, dx + F_y \, dy \]
\[ = \int_0^1 4(2x) \, dx - 2x(2dx) \]
\[ = \int_0^1 4x \, dx = 2x^2 \bigg|_0^1 = 2 \text{ (Joules if } \overrightarrow{F} \text{ in Newton)} \]
\[ \text{and } x, y \text{ in meters} \]

\[ 5) \quad y = 2x^2 \]
\[ dy = 4xdx \]
\[ \int \overrightarrow{F} \cdot dr = \int_0^1 4(2x^2) \, dx - 2x \cdot (4xdx) \]
\[ = \int_0^1 4x \, dx = 0. \]

The works along the two paths are not equal.

That’s okay because \( \overrightarrow{F} \) is not conservative

\[ \frac{\partial F_y}{\partial x} = -2 \quad \frac{\partial F_x}{\partial y} = 4 \]
It is interesting to ask why $W=0$ along path b.

\[ \vec{F} = 4y\hat{i} - 2x\hat{j} = 8x^2\hat{i} - 2x\hat{j} \]

\[ = 2x(4\hat{i} - \hat{j}) \]

and \[ \vec{dr} = dx\hat{i} + dy\hat{j} \]

\[ = dx(\hat{i} + 4x\hat{j}) \]

You can see that $\vec{F}$ and $\vec{dr}$ are always \perp to each other along path b! Hence $W=0$. 
Writing $z = e^{i\theta}$ we have

$$\cos \theta = \frac{(e^{i\theta} + e^{-i\theta})}{2} \quad \text{and} \quad dz = i \, e^{i\theta} \, d\theta$$

$$= \left( z + \frac{1}{z} \right) \frac{1}{2} \quad \frac{d\theta}{dz} = \frac{dz}{i}$$

Also as $\theta$ goes from $\phi \to 2\pi$, $z$ follows a circular path around the origin.

$$I = \oint \frac{dz}{z^2 + 5z + 2} = \oint \left( \frac{1}{z - 2 - \frac{1}{2}} \right)$$

$$= \oint \left( \frac{1}{z - 2 - \frac{1}{2}} \right) \frac{dz}{iz} = \oint \frac{dz}{z^2 + 5z + 2}$$

$$z = \frac{-5 \pm \sqrt{5^2 - 4(2)(2)}}{2(2)} = \frac{-5 \pm \sqrt{9}}{4}$$

$$z = -2, \quad z = -\frac{1}{2}$$

Can see this also by factorizing:

$$2z^2 + 5z + 2 = (2z + 1)(z + 2)$$

The pole at $z = -\frac{1}{2}$ is inside the circle and has residue

$$R = \lim_{z \to -\frac{1}{2}} \left( z + \frac{1}{2} \right) \frac{1}{(2z + 1)(z + 2)}$$
\[ R = \lim_{2 \to 3} \frac{1}{2} \frac{1}{2 + 2} = \frac{1}{3} \]

So from Residue theorem,

\[ I = \frac{1}{i} 2\pi i \left( \frac{1}{3} \right) = \frac{2\pi}{3} \cdot i. \]

Check this pictorially:

\[
\frac{1}{5 + 4 \cos \theta}
\]

\[
\frac{1}{\theta}
\]

Seems like \( I \) < area under ---

\[ I < \frac{1}{2} (2\pi) 1 = \pi \]

But probably bigger than area under ----

\[ I > \frac{1}{2} \pi 1 = \frac{\pi}{2} \]

So \( \frac{2\pi}{3} \) is reasonable.
(a) \[ e^{i\theta} = \cos \theta + i\sin \theta \]

(b) A proof comes from using Taylor series for \( \sin, \cos, \) and exponential. Actually doing it:

\[ e^x = 1 + x + x^2/2 + x^3/6 + \ldots \]

\[ e^{i\theta} = 1 + i\theta - \theta^2/2 - i\theta^3/6 \ldots \]

\[ = (1 - \theta^2/2 + \ldots) + i(\theta - \theta^3/6 + \ldots) \]

\[ \cos \theta \quad \sin \theta \]

(c) \[ e^{2i\theta} = \cos 2\theta + i\sin 2\theta \]

\[ e^{2i\theta} = (e^{i\theta})(e^{i\theta}) = (\cos \theta + i\sin \theta)^2 \]

\[ = \cos^2 \theta - \sin^2 \theta + 2i\sin \theta \cos \theta \]

So \[ \cos 2\theta = \cos^2 \theta - \sin^2 \theta \]

\[ \sin 2\theta = 2\sin \theta \cos \theta \]
In a random walk of $N$ steps with $p = q = \frac{1}{2}$, the most likely outcome is to end where you begin (at the origin). We know that other outcomes are possible, but are reasonably likely only if $\sqrt{x^2} \sim \sqrt{N}$.

Applying that here, $N = 4,000,000$ and $\sqrt{N} = 2,000$.

We'd expect 9.11 or $2,000,000 \pm 2000$\footnote{vote difference is "expected"},

\[ \text{Results} = 2,000,000 \pm 2000 \]

but a 100,000 vote difference is just impossible!

You could actually work it out with Stirling's formula

\[ \ln N! = N \ln N - N + \frac{1}{2} \ln 2\pi N \]

\[
\begin{array}{c|c}
\text{deviation} & \text{ratio of prob to most likely outcome (tie)} \\
100 & .9950 \\
200 & .9862 \\
500 & .9825 \\
1000 & .9665 \\
2000 & .9353 < e^{-12.5} \\
5000 & .9000 < e^{-50} \\
10,000 & .8500 < e^{-200} \\
20,000 & .8000 \\
\end{array}
\]