[1.] Write down the solutions to the equation \( z^6 = 2 \).

[2.] In understanding the behavior of spin-1 particles in quantum mechanics, you will encounter the \( 3 \times 3 \) matrix

\[
S_z = \frac{\hbar}{\sqrt{2}} \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]

What are the possible values you could get if you measure the \( x \) component of spin of a spin-one particle in an experiment? If your system is in the state \( \tilde{\psi} = (1, 0, 0) \), what are the probabilities of measuring the different possible values of \( S_z \)?

[3.] What is the definition of a projection operator?
Prove that the eigenvalues of a projection operator must be \( \lambda = 0, 1 \).

[4.] Use binomial theorem (left equation below) to evaluate the sum \( S \) on the right.

\[
(p + q)^N = \sum_{j=0}^{N} \binom{N}{j} p^j q^{N-j} \quad \quad \quad S = \sum_{j=0}^{N} \binom{N}{j} j p^j q^{N-j}
\]

[5.] Two equal masses \( m \) are connected to each other by a spring of force constant \( k \). One of the masses is also connected to a "wall" (ie a completely stationary object) by a spring of force constant \( 2k \).
(a) Write down the equations of motion \( F = ma \) for the two masses.
(b) Assume a solution \( x_1 = v_1 e^{i\omega t} \) for \( l = 1, 2 \). What equations do \( v_1 \) and \( v_2 \) obey?
(c) Compute the normal mode frequencies.
(d) Compute the normal mode vectors.
(e) In previous problems we have always found \( \omega = 0 \) as one of the frequencies. Did that happen here? Give a physical argument why still having \( \omega = 0 \) makes sense, or why you do not expect \( \omega = 0 \), depending on your answer.
1. \( z^6 = 2 \)

Write \( z = r e^{i\theta} \)

Then \( r^6 e^{6i\theta} = 2 \)

We need \( r = \sqrt[6]{2} \approx 1.1225 \)

and \( 6\theta = 2\pi n \quad n = 0, 1, 2, 3, 4, 5 \)

so \( \theta = \frac{\pi}{3} \left\{ 0, 1, 2, 3, 4, 5 \right\} \)

Pictorially:

...
Possible values of measured are eigenvalues of matrix representing $S_\alpha$

\[
0 = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda (\lambda^2 - 1) + 1(-\lambda) 
\]

\[
\lambda (\lambda^2 - 2) = 0 
\]

\[
\lambda = \{-\sqrt{2}, \sqrt{2}\} \quad \lambda \sqrt{2} = \{-1, 0, 1\} \quad k 
\]

Eigenvectors are, for $\lambda = -\sqrt{2}$

\[
\begin{pmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} 
\]

\[
\begin{pmatrix} 1 & -\sqrt{2}/2 & 0 \\ 0 & -\sqrt{2}/2 & 1 \\ 0 & 1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & -\sqrt{2}/2 & 0 \\ 0 & 1 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} 
\]

$v_1 + \sqrt{2}/2 v_2 = 0 
\]

$v_1 = -\sqrt{2}/2 v_2 
\]

$v_2 + \sqrt{2} v_3 = 0 
\]

$v_3 = -\sqrt{2}/2 v_2 
\]

\[
\begin{pmatrix} -\sqrt{2}/2 \\ -1 \\ -\sqrt{2}/2 \end{pmatrix} \Rightarrow \begin{pmatrix} -1/2 \\ -\sqrt{2}/2 \\ -1/2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}
\]
2-2

For $\lambda = 0$

$$
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix} \rightarrow 
\begin{pmatrix}
1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
$$

$$
\Rightarrow 
V_1 = -V_3 \\
V_2 = 0 \\
\frac{1}{\sqrt{2}} \begin{pmatrix}
1 \\
0 \\
-1
\end{pmatrix} = \phi_2
$$

And for $\lambda = \pm \sqrt{2}$

$$
\begin{pmatrix}
-\sqrt{2} & 1 & 0 \\
1 & -\sqrt{2} & 1 \\
0 & 1 & -\sqrt{2}
\end{pmatrix} \rightarrow 
\begin{pmatrix}
1 & -\sqrt{2} & 0 \\
0 & -\sqrt{2} & 1 \\
0 & 0 & 0
\end{pmatrix} \rightarrow 
\begin{pmatrix}
1 & -\sqrt{2} & 0 \\
0 & 1 & -\sqrt{2}
\end{pmatrix}
$$

So that

$$
V_1 - \sqrt{2}/2 V_2 = 0 \\
V_1 = \frac{\sqrt{2}}{2} V_2 \\
V_2 - \sqrt{2} V_3 = 0 \\
V_3 = \frac{\sqrt{2}}{2} V_2
$$

$$
\frac{1}{\sqrt{2}} \begin{pmatrix}
\sqrt{2}/2 \\
1 \\
1/2
\end{pmatrix} \rightarrow 
\begin{pmatrix}
1/2 \\
\sqrt{2}/2 \\
1/2
\end{pmatrix} = \phi_3
$$

One can verify the $\{ \phi_i \}$ are $\perp$ as expected.

Writing

$$
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} = q_1 \frac{1}{2} \begin{pmatrix}
-1 \\
\sqrt{2} \\
-1
\end{pmatrix} + q_2 \frac{1}{\sqrt{2}} \begin{pmatrix}
1 \\
0 \\
-1
\end{pmatrix} + q_3 \frac{1}{\sqrt{2}} \begin{pmatrix}
1 \\
0 \\
1
\end{pmatrix}
$$

Yields

$$
1 = -q_1/2 + q_2/\sqrt{2} + q_3/\sqrt{2}
$$

$$
0 = -q_1 \sqrt{2}/2 + q_3 \sqrt{2}/2
$$

$$
0 = -q_1/2 - q_2/\sqrt{2} + q_3/2
$$
Thus \( q_y = -q_1 \), and

\[ 0 = -q_1/2 - q_2/\sqrt{2} - q_1/2. \]

So that \( q_2 = -\sqrt{2} q_1 \).

And \( 1 = -q_1/2 - q_1 - q_1/2 \)

\[ q_1 = -\frac{1}{2}, \quad q_2 = \frac{\sqrt{2}}{2}, \quad q_3 = \frac{1}{2}. \]

Prob of measuring \( -\frac{1}{\hbar} \) \( \Rightarrow \) \( |q_1|^2 = \frac{1}{4} \)

\( 0 \) \( \Rightarrow \) \( |q_2|^2 = \frac{1}{2} \)

\( +\frac{1}{\hbar} \) \( \Rightarrow \) \( |q_3|^2 = \frac{1}{4} \)
A projection operator obeys $P^2 = P$

If $PV = \lambda V$ applying $P$ to both sides

then $P^2V = \lambda PV$

$PV = \lambda^2 V$ using $P^2 = P$ and $PV = \lambda V$

$\lambda V = \lambda^2 V$ using $PV = \lambda V$

Thus $\lambda^2 = \lambda$ so that $\lambda = 0$ or $\lambda = 1$
\[(p+q)^n = \sum_{j=0}^{n} \binom{n}{j} p^j q^{n-j} \]

Differentiate with respect to \(p\)

\[N(p+q)^{n-1} = \sum_{j=0}^{n} \binom{n}{j} j p^{j-1} q^{n-j} \]

Multiply by \(p\)

\[N (p+q)^{n-1} p = \sum_{j=0}^{n} \binom{n}{j} j p^j q^{n-j} \]

so \[S = Np (p+q)^{n-1} \]

often we encounter \(p+q = 1\) in

which case \(S' = Np\)
\[ \text{mass 1} \quad \downarrow \quad \text{mass 2} \]

\[ 2k \quad k \]

\[ \begin{array}{c}
\text{mass 1} \\
\downarrow \\
m
\end{array} \quad \begin{array}{c}
\downarrow \\
m
\end{array} \quad \begin{array}{c}
\text{mass 2} \\
m
\end{array} \]

\[ \begin{align*}
a) \quad m \frac{d^2x_1}{dt^2} &= -2k x_1 - k (x_1 - x_2) \\
m \frac{d^2x_2}{dt^2} &= -k (x_2 - x_1)
\end{align*} \]

\[ b) \quad -m \omega^2 v_1 = -2k v_1 - k (v_1 - v_2) \]

\[ -m \omega^2 v_2 = -k (v_2 - v_1) \]

\[ (3k - m\omega^2) v_1 - k v_2 = 0 \]

\[ -k v_1 + (k - m\omega^2) v_2 = 0 \]

\[ 4) \quad (3k - m\omega^2)(k - m\omega^2) - k^2 = 0 \]

\[ (m\omega^2)^2 - 4k m\omega^2 + 3k^2 - k^2 = 0 \]

\[ (m\omega^2) = \left[ 4k \pm \sqrt{16k^2 - 4(2k^2)} \right] / 2 \]

\[ = (2 \pm \sqrt{2})k \]

\[ \omega^2 = (2 \pm \sqrt{2})^2 / m \]
\[
(d) \quad mw^2 = (2 + \sqrt{2}) k
\]

\[
(1 - \sqrt{2}) v_1 - v_2 = 0 \quad \Rightarrow \quad \text{this eqn is just times the top eqn}
\]

\[
= \frac{1}{1 - \sqrt{2}} \begin{pmatrix} 1 \\ 1 - \sqrt{2} \end{pmatrix}
\]

\[
\phi_1 = \begin{pmatrix} 1 \\ 1 - \sqrt{2} \end{pmatrix}
\]

\[
(1 + \sqrt{2}) v_1 - v_2 = 0 \quad \Rightarrow \quad \text{this eqn is just times the top eqn}
\]

\[
= \frac{1}{1 + \sqrt{2}} \begin{pmatrix} 1 + \sqrt{2} \\ 1 \end{pmatrix}
\]

\[
\phi_2 = \begin{pmatrix} 1 + \sqrt{2} \\ 1 \end{pmatrix}
\]

\[
mw^2 = (2 - \sqrt{2}) k
\]

\[
(1 + \sqrt{2}) v_1 - v_2 = 0 \quad \Rightarrow \quad \text{this eqn is just times the top eqn}
\]

\[
\phi_2 = \begin{pmatrix} 1 + \sqrt{2} \\ 1 \end{pmatrix}
\]

\[
\phi_1 \perp \phi_2 \quad \text{as expected since we have a real, symmetric matrix.}
\]
Picture \( \vec{\phi}_1 = \begin{pmatrix} 1 \\ -0.414 \end{pmatrix} \) \( \vec{\phi}_2 = \begin{pmatrix} 1 \\ 2.414 \end{pmatrix} \)

\[
\begin{array}{c}
| \rightarrow \leftarrow \| \\
\| m-o-m-o \|
\end{array}
\quad
\begin{array}{c}
| \rightarrow \rightarrow \| \\
\| m-o-m-o \|
\end{array}
\]

higher \( \omega^2 \) \( (2+\sqrt{2}) \text{ kNm}^{-2} \)

lower \( \omega^2 \) \( (2-\sqrt{2}) \text{ kNm}^{-2} \)

we do not get \( \omega = 0 \) because the forces are not all 'internal' i.e. part of action-reaction pairs. There is an external spring which keeps the system tethered to the wall. We cannot do a uniform translation at no energy cost.