Applications of Fourier Series

1. To quantum mechanics.

Schrödinger eqn

\[ i\hbar \frac{\partial}{\partial t} \psi(x,t) = \hat{H} \psi(x,t) \]

\[ \psi(x,t) = e^{-i\hat{H}t/\hbar} \psi(x,0) \]

\[ \exp \quad \psi(x,0) = \sum q_n \phi_n(x) \]

Consider a square well:

\[ \psi(x,t) = \sum q_n e^{-iE_n t/\hbar} \phi_n(x) \]

Operator \( \hat{H} \) replaced by number \( E_n \).

\[ \psi(x,0) = A x (L-x) \]

Diagram of a potential well with parabola max at \( L/2 \).
First, normalize \( \psi(x, 0) = A \cdot (L - x) \)

\[
1 = \int_0^L |\psi(x, 0)|^2 \, dx = \int_0^L A^2 x^2 (L - x)^2 \, dx \\
= A^2 \int_0^L (x^2 L^2 - 2Lx^3 + x^4) \, dx \\
= A^2 \left[ \frac{x^3 L^2}{3} - \frac{L^4 x}{4} + \frac{x^5}{5} \right]_0^L \\
= A^2 \left( L^5 \left( \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \right) \right) = A^2 \frac{17L^5}{60}
\]

\[
A = \frac{\sqrt{60}}{\sqrt[5]{17L^5}}
\]

Next, determine \( \phi_n(x) \) and \( E_n \).

Inside the infinite square well \( V(x) = 0 \)

\[
\hat{H} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2}
\]

\[
\hat{H} \phi_n(x) = E_n \phi_n(x)
\]

has solutions \( \phi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{nk}{L} \)

\[
E_n = \frac{\hbar^2 k^2}{2m}
\]

normalized

\[
k = \frac{n \pi}{L}
\]

guarantees \( \phi_n(L) = 0 \)
We now recognize our expansion of \( f(x,0) \) in

\[
f_m(x) \quad \text{as just a Fourier series}
\]

\[
A x (L-x) = \sum_{n=1}^{\infty} a_n \sqrt{\frac{2}{L}} \sin k_n x
\]

Multiply by \( \sqrt{\frac{2}{L}} \sin k_m x \) and integrate

\[
\int_0^L A x (L-x) \sin \frac{m \pi x}{L} \, dx = q_m
\]

You learned how to do this integral in Math 21

by integrating by parts using sine, cosine. It is

somewhat easier using complex exponentials

Using results of pages AFS-3A, B

\[
q_m = \frac{1}{L} \int_0^L A x (L-x) \sin \frac{m \pi x}{L} \, dx
\]

\[
q_m = \frac{1}{L} \left[ -\frac{L}{m \pi} \cos \frac{m \pi x}{L} + \left( \frac{L}{m \pi} \right)^2 \sin \frac{m \pi x}{L} \right]_0^L
\]

\[
- A \left\{ -\frac{L}{m \pi} \cos \frac{m \pi x}{L} + \left( \frac{L}{m \pi} \right)^2 x \sin \frac{m \pi x}{L} + 2 \left( \frac{L}{m \pi} \right)^3 \cos \frac{m \pi x}{L} \right\}_0^L
\]
\[ \int x \sin kx \, dx \]

\[ = \text{Im} \left[ \int \frac{x e^{ikx}}{u} \, du \right] \]

\[ = \text{Im} \left\{ \frac{x e^{ikx}}{ik} - \int \frac{e^{ikx}}{(ik)^2} \, dx \right\} \]

\[ = \text{Im} \left\{ -\frac{i}{k} x e^{ikx} - \frac{e^{ikx}}{(ik)^2} \right\} \]

\[ e^{ikx} = \cos kx + i \sin kx \]

\[ = -\frac{x \cos kx}{k} + \frac{\sin kx}{k^2} \]

Check by differentiating

\[ \frac{d}{dx} \left( -\frac{x \cos kx}{k} + \frac{\sin kx}{k^2} \right) = -\frac{\cos kx}{k} + x \sin kx + \frac{\cos kx}{k^2} \]
Similarly, \( \int x^2 \sin kx \, dx \)

\[
= \text{Im} \left[ \int x^2 e^{ikx} \, dx \right]
= \text{Im} \left[ \int x^2 e^{ikx} \, du \right]
= \text{Im} \left[ \int x^2 \frac{e^{ikx}}{ik} \, dx \right] - \int \frac{e^{ikx}}{ik} \cdot 2x \, dx
\]

\[
= \text{Im} \left\{ -\frac{i x^2}{k} e^{ikx} + \frac{2i}{k} \int xe^{ikx} \, dx \right\}
= -\frac{x^2}{k} \cos kx + \frac{2}{k} \text{Re} \left[ \int xe^{ikx} \, dx \right]
= -\frac{x^2}{k} \cos kx + \frac{2}{k} \left\{ \frac{x}{k} \sin kx + \frac{\cos kx}{k^2} \right\}
\]

from AFS-3A

\[
= -\frac{x^2}{k} \cos kx + \frac{2x}{k^2} \sin kx + \frac{2\cos kx}{k^2}
\]

Check by differentiation:

\[
-\frac{2x}{k} \cos kx + x^2 \sin kx + \frac{2}{k} \sin kx + \frac{2x}{k^2} \cos kx - \frac{2\sin kx}{k}
\]
\[ \psi_m = A_L \left\{ -\frac{L^2}{m \pi} (-1)^m \right\} \]

\[- A \left\{ -\frac{L^3}{m \pi} (-1)^m + 2 \left( \frac{L}{m \pi} \right)^3 (-1)^m - 1 \right\} \]

Terms 1 and 2 cancel, leaving

\[ \psi_m = \sqrt{\frac{2}{L}} g_A \left( \frac{L}{m \pi} \right)^3 \left\{ 1 - (-1)^m \right\} \]

This is nonzero only for \( m = 1, 3, 5, \ldots \)

\[ \text{as expected, also falls off rapidly with } m. \]

This is because the parabola looks almost like \( m = 1 \) term itself!

\[ m = 2 \text{ -- wrong symmetry!} \]

Final result:

\[ \psi(x,t) = \sum_{m=odd} 4 A \left( \frac{L}{m \pi} \right)^3 e^{-i \frac{-k^2}{2} \frac{2 \pi^2 t}{L^2}} \left( \frac{m \pi x}{L} \right) \]

\[ A = \sqrt{60/17L^5} \]

Check units: Expect \( \psi \sim \frac{1}{\sqrt{L}} \), and it is \( L^3 \sqrt{L^{5/2}} \), \( \checkmark \)
(2) To wave equation

If you stretch a string from equilibrium

\[ \frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} = 0 \]

(Note: Many other examples e.g. pressure in a pipe)

What is \( v \)?

A: \( v = \sqrt{\frac{T}{\mu}} \)

\( T = \) tension

\( \mu = \) mass/length

What other examples do you know?

A: EM waves!
Does anyone recall derivation of wave eqn for
vibrating string?

piece of string mass \( m dx \)

\[ T \frac{dy}{dx} \bigg|_x \]

and at \( x + dx \)

\[ T \frac{dy}{dx} \bigg|_{x+dx} \]

For any \( f(x + dx) = f(x) + \frac{df}{dx} \bigg|_x \ dx \)

apply to \( f = \frac{dy}{dx} \)

\[ \frac{dy}{dx} \bigg|_{x+dx} = \frac{dy}{dx} \bigg|_x + \frac{d^2y}{dx^2} \bigg|_x \ dx \]

so net force in \( y \) direction is

\[ T \ dx \frac{d^2y}{dx^2} = m \ dx \ \frac{d^2y}{dx^2} \]

\[ \text{Fnet} = \ m \ \frac{d^2y}{dx^2} \]

\[ \frac{d^2y}{dx^2} = \frac{1}{(T/m)} \ \frac{d^2y}{dt^2} \]
Does anyone recall derivation of wave equation in E&M? This is trickier. First, wave eqn in 3D is

\[ \nabla^2 y(x,t) - \frac{1}{\sqrt{c^2}} \frac{\partial^2 y(x,t)}{\partial t^2} = 0 \]

Laplacean \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \)

Then use Maxwell eqns

\[ \vec{\nabla} \cdot \vec{B} = 0 \]

\[ \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} = 0 \quad \text{in free space} \]

\[ \vec{\nabla} \times \vec{B} = \mu_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \]

\[ \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]

and a vector identity

\[ \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = -\vec{\nabla} (\vec{\nabla} \cdot \vec{A}) + \nabla^2 \vec{A} \]

\[ \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{B}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \]

\[ \vec{\nabla} (\vec{\nabla} \cdot \vec{E}) + \nabla^2 \vec{E} \]

\[ \nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \]
Separation of variables

\[ y(x,t) = f(x)g(t) \]

\[ f''(x)g(t) = \frac{1}{\sqrt{2}} f(x) g''(t) \]

\[ \frac{f''(x)}{f(x)} = \frac{1}{\sqrt{2}} \frac{g''(t)}{g(t)} \]

Function of \( f(x) \)

Function of \( g(t) \)

The only way this can happen is if both are constants

\[ f''(x) = -k^2 = \frac{1}{\sqrt{2}} \frac{g''(t)}{g(t)} \]

\[ f(x) = \sin kx \cos kx \quad \text{(usual solution)} \]

\[ g(t) = \sin kvt \cos kvt \quad (\text{e.g. for right-sided, } w = kv, \quad w = ck) \]

If we insist \( y(0,t) = 0 \) (string fixed at ends)

\[ y(L,t) = 0 \quad \text{deem at ends} \]

Just as in our QM problem we have only

\[ f(x) = \sin \frac{m \pi x}{L} \quad \text{as possible } f(x) \]
\[ \cos kx \text{ eliminated by } y(0, t) = 0 \]

and \[ k \to m\pi x / L \text{ only by } y(L, t) = 0 \]

\[ y(x, t) = \sum_{m=1}^{\infty} \sin \frac{m \pi x}{L} \left\{ a_m \cos \frac{m \pi v t}{L} + b_m \sin \frac{m \pi v t}{L} \right\} \]

If the string starts from rest \[ \frac{\partial y}{\partial t} = 0 \text{ at } t = 0 \]

then all the \( b_m = 0 \) because

\[ \left. \frac{\partial y}{\partial t} \right|_{t=0} = \sum_{m=1}^{\infty} \sin \frac{m \pi x}{L} \left\{ \frac{m \pi v}{L} b_m \right\} \]

Solving \[ y(x, t) \]

from rest

\[ y(x, 0) = \sum_{m=1}^{\infty} a_m \sin \frac{m \pi x}{L} \]

\[ y(x, 0) = \sum_{m=1}^{\infty} a_m \sin \frac{m \pi x}{L} \left( \text{multiply by } \frac{2}{L} \int_0^L y(x, 0) \sin \frac{m \pi x}{L} \, dx = a_m \right) \]

and integrate:
Starting from rest

So solving wave eqn for vibrating string is

practically identical to Schrödinger for 1D square well

\[ U(x,0) = \sum_m q_m \sqrt{\frac{2}{L}} \sin \frac{n \pi x}{L} \]
\[ y(x,0) = \sum_m q_m \sin \frac{n \pi x}{L} \]

Determine \( q_m \)

\[ q_m = \sqrt{\frac{2}{L}} \int_0^L U(x,0) \sin \frac{n \pi x}{L} \, dx \]
\[ q_n = \frac{2}{L} \int_0^L y(x,0) \sin \frac{n \pi x}{L} \, dx \]

\[ U(x,t) = \sum_{n=1}^\infty q_n e^{-\frac{En t}{\hbar}} \sqrt{\frac{2}{L}} \sin \frac{n \pi x}{L} \]
\[ y(x,t) = \sum_{n=1}^\infty q_n \sin \frac{n \pi x}{L} \cos \frac{n \pi vt}{L} \]

Can make look even more similar via

\[ \cos \omega_n t \]
\[ \omega_n = \sqrt{\frac{n \pi \hbar}{L}} \]

\[ E_n = \frac{k^2}{2m} \frac{n^2 \pi^2 \hbar^2}{L^2} \]
Really Fancy Stuff are GREEN'S FUNCTIONS

\[ y(x,t) = \sum_{m} q_m \sin \frac{m \pi x}{L} \cos \frac{m \pi \nu t}{L} \]

\[ q_m = \frac{2}{L} \int_{0}^{L} \sin \frac{m \pi x}{L} y(x,0) \, dx \]

\[ y(x,t) = \sum_{m} \left( \int_{0}^{L} \frac{2}{L} \sin \frac{m \pi x}{L} y(x',0) \, dx' \sin \frac{m \pi \nu t}{L} \cos \frac{m \pi x}{L} \right) \]

\[ = \int_{0}^{L} dx' y(x',0) \left( \sum_{m} \frac{2}{L} \sin \frac{m \pi x}{L} \sin \frac{m \pi \nu t}{L} \cos \frac{m \pi x}{L} \right) \]

This is \( G(x,x',t) \)
the "green's function"
of the wave eqn.

\[ y(x,t) = \int_{0}^{L} dx' y(x',0) G(x,x',t) \]

\( G \) describes spread of information from knowledge at position \( x \) to \( x' \) at time \( t \).
Example 3: The driven LRC circuit

(C or driven mass on spring ↔ any oscillator)

\[ L \frac{d^2 \theta}{dt^2} + R \frac{d\theta}{dt} + \frac{\theta}{C} = 0 \]

\[
\begin{align*}
\text{Suppose} & \quad V(t) = V_0 \cos \omega t \\
Q(t) & = R e^{-i \omega t} + \frac{V_0 e^{i \omega t - \phi}}{\sqrt{L^2 (\omega_0^2 - \omega^2) + \omega^2 R^2}}
\end{align*}
\]

\[ \tan \phi = \frac{\omega R}{L (\omega_0^2 - \omega^2)} \]

\[ \omega_0^2 = \frac{1}{LC} \]

We did this problem in lecture 1.

But what about general driving functions like

\[
\begin{align*}
V(t) & \quad \text{Graph of the driving function}
\end{align*}
\]

How do we compute the response of the circuit?
The key is to take advantage of the fact that the differential eqn is linear. Thus its response to \( V_1(t) + V_2(t) \)

can be obtained as \( Q_1(t) + Q_2(t) \)

where \( Q_1 \) and \( Q_2 \) are responses to \( V_1 \) and \( V_2 \)

So what needs to be done is to write the periodic function \( V(t) \)

as a Fourier expansion

\[ \sin \frac{\pi t}{T} \quad \text{and} \quad \cos \frac{\pi t}{T} \]

This was basically Example #1 of Matalay's lecture!

Q: Does anyone recall some of basic result?

A: \( \cos \frac{\pi t}{T} \) terms are absent since \( V(t) \) is odd

**SUMMARY:** Almost any where you look, solving physics problems involves Fourier series.

- \( QM, CM, EM, \) EM waves or LRC circuit
- particle
- wave
- vibrating string or mass/spring good fit