Fourier Integrals and Dirac Delta Function

Anatoly introduced the complex form of Fourier series.

\[ f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{n \pi x}{L}} \]

\[ c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i \frac{n \pi x}{L}} dx \]

Integers.

We have discussed how the collection of continuous functions on an interval forms a vector space and the numbers that \( f(x) \) can be viewed as the components of \( \mathbf{f} \).

Commenting on analogies like

\[ \mathbf{\hat{v}} \cdot \mathbf{\hat{w}} = v_1 \hat{w}_1 + v_2 \hat{w}_2 + v_3 \hat{w}_3 \]

\[ \Leftrightarrow f \cdot g = \int_{dx} f(x) g(x) \quad \text{etc.} \]

We also noted that the Fourier coefficients can be viewed as the components of \( f \) in the new basis of \( \sin \frac{n \pi x}{L}, \cos \frac{n \pi x}{L} \) (or \( e^{i n \pi x/L} \)) which form a complete set.
There is something a bit odd about this.

For conventional vectors in real space we have

3 components and if we change basis we still have

3 components. Here in our original basis $f(x)$ has a continuous infinity of components while

in the new Fourier basis a discrete infinity $c_n \in \mathbb{Z}$.

This asymmetry is reflected in the transformation $\mathcal{F}[f(x)] \rightarrow (c_n)$ which look dissimilar:

This lack of symmetry goes away if we

consider the limit $L \rightarrow \infty$ (and hence consider general

functions $f$, not just periodic ones).
Let's first note that in the relations between $f(x)$ and $c_n$ we can place the prefactors wherever we like:

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{i\pi n/L} x$$

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x)e^{-i\pi n/L} \, dx$$

$$\leftrightarrow f(x) = \frac{1}{2L} \sum_{-\infty}^{\infty} c_n e^{i\pi n/L} x$$

$$c_n = \int_{-L}^{L} f(x)e^{-i\pi n/L} \, dx$$

Here $c_n$ will be $2L$ times bigger.

Compensated by dividing by $2L$ here.

Let's also introduce the notation $k = \frac{\pi n}{L}$

$$f(x) = \frac{1}{2L} \sum_{-\infty}^{\infty} c_k e^{ikx} x$$

$$c_k = \int_{-L}^{L} f(x)e^{-ikx} \, dx$$

Finally, recall the way we get integrals as a sum ("rectangle rule")

$$\int_{a}^{b} g(u) \, du = \sum_{n=1}^{N} g_0 e^{\Delta y}$$

Values of $g$ at discrete points

$$\sum_{n=0}^{N} g_n = \frac{1}{\Delta y} \int_{a}^{b} g(u) \, du$$
As \( L \to 0 \) our \( k \) points get more and more closely spaced \( \varepsilon = \frac{\pi}{L} \). The \( k \) integral will become a good approximation:

\[
f(x) = \frac{1}{2L} \frac{L}{\pi} \int_{-\infty}^{\infty} c(k) e^{ikx} \, dk
\]

**This is called a "Fourier integral".** These

are look very similar to each other, emphasizing the \( f(x) \) and \( c(k) \) bases are really the same thing, sometimes people even write

\[
f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c(k) e^{-ikx} \, dk \quad c(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx
\]

to make it look really identical.
Early in the course we noted that when you sum up all the roots of 1 you get zero:

\[ \sum_{n=-\frac{L}{2}+1}^{\frac{L}{2}} e^{\frac{2\pi i}{L} n} = 0 \]

\( L = 6 \)

\( n = 2 \)  \( n = 1 \)  \( n = 0 \)  \( n = -1 \)  \( n = -2 \)

It is also true that if you raise these roots to the power \( p \) and add them, you get zero. e.g. for \( p = 2 \):

\[ \sum_{n=-\frac{L}{2}+1}^{\frac{L}{2}} (e^{\frac{2\pi i}{L} n})^p = L \delta_{p,0} \]

Kronecker delta
If we define \( k = \frac{2\pi n}{L} \) then tells us

\[ \sum_{k} e^{ikp} = L \delta_{p,0} \]

and taking \( L \to 0 \) and converting to integral as before

\[ \frac{1}{2\pi} \int_{-\infty}^{0} dk \, e^{ikp} = L \delta_{p,0} \]

\[ \frac{1}{2\pi} \text{ factor} \]

\[ \int_{-\infty}^{0} dk \, e^{ikp} = 2\pi \delta_{p,0} \]

\[ \int_{-\infty}^{\infty} dx \, e^{ikx} = 2\pi \delta(x) \]
The Fourier integral plays a particularly central role in QM. Real space wave function \( \psi(x) \):

\[
|\psi(x)|^2 \, dx = \text{prob qm particle between } x \text{ and } x + dx
\]

\[
\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = 1
\]

If \( \varphi(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \psi(x) \, dx \)

\[
\int \varphi(p)^2 \, dp = \text{prob qm particle has momentum between } p \text{ and } p + dp
\]

Consider \( \psi(x) = \frac{\sqrt{2}}{\sqrt{L}} \sin \frac{\pi x}{L} \)

\[
\text{prob } |\psi(x)|^2 = \frac{2}{L} \sin^2 \frac{\pi x}{L}
\]

sort of wave shape

Q: Why?
A: \( \sin x \sim x \), \( \sin^2 x \sim x^2 \)
Q: What is probability particle has momentum $p$?

Any guesses?

$$
\phi(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{L}} \sin \frac{\pi x}{L} e^{-i \frac{px}{\hbar}} dx
$$

$$
= \frac{1}{\sqrt{2\pi}} \int_{0}^{L} \sqrt{\frac{2}{L}} e^{\frac{i\pi x}{L}} e^{-\frac{i\pi x}{L}} e^{\frac{ipx}{\hbar}} dx
$$

$$
= \frac{1}{\sqrt{2\pi}} \frac{1}{2} \sqrt{\frac{2}{L}} \int_{0}^{L} (e^{\frac{i(\pi/L + p/\hbar)x}{L}} - e^{\frac{i(-\pi/L + p/\hbar)x}{L}}) dx
$$

$$
\frac{e^{\frac{i\pi L}{L} - 1}}{-1} - \frac{e^{\frac{i\pi L}{L} + p/\hbar} - 1}{i(\pi/L + p/\hbar)}
$$

$$
= \frac{1 + e^{ipL/\hbar}}{iL} \left\{ \frac{\pi/L - p/\hbar + \pi/L + p/\hbar}{(\pi/L + p/\hbar)(-\pi/L + p/\hbar)} \right\}
$$

$$
= \frac{2\pi}{iL} \frac{1}{1 + e^{ipL/\hbar}} \frac{1}{p^2/\hbar^2 - \pi^2/L^2}
$$
\[ |\psi(p)|^2 = \frac{2\pi}{L} \left( \frac{p^2 - \pi^2}{L^2} \right)^2 \frac{(1 + e^{i p L / L})(1 - e^{-i p L / L})}{2(1 + \cos p L / L)} \]

peaked at \( p = \frac{\pi}{L} \)

as expected!
SERIES

\[ f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx\pi/L} \quad \quad \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \ c(k) e^{-ikx} \]

\[ c_n = \frac{1}{L} \int_{-L}^{L} f(x) e^{-inx\pi/L} \ dx \quad \quad \quad c(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \ f(x) e^{-ikx} \]

\[ \int_{-L}^{L} e^{inx\pi/L} e^{-inx\pi/L} \ dx = 2L \delta_{n,m} \quad \quad \quad \int_{-\infty}^{\infty} e^{i(k-k')x} \ dx = 2\pi \delta(k-k') \]

INTEGRAL

Then \[ \int_{-L}^{L} \ dx \ f(x) = 2L \delta_{nm} \quad \quad \quad \int_{-\infty}^{\infty} dx \ f(x) e^{-ik'x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \ c(k) e^{-ik'x} \]
Applications of Fourier Integrals - II

1. QM \( \psi(x,0) = \eta e^{-x^2/2\Delta^2} \) \( \Rightarrow \) start with wave function which is Gaussian of width \( \Delta \)

\[
\int |\psi(x,0)|^2 dx = \eta^2 \int_{-\infty}^{\infty} e^{-x^2/2\Delta^2} dx = \eta^2 \sqrt{\pi \Delta^2} = \eta^2 \sqrt{\pi}
\]

Thus \( \eta = \frac{1}{\sqrt{\Delta \pi}} \). Dimensions \( \frac{\Delta}{\pi} \) has units of length and \( \psi \) of \( \frac{1}{\sqrt{\pi} \Delta} \)

Free particle \(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x,t) = -\hbar \frac{\partial \psi(x,t)}{\partial t}\)

\( \psi(x,t) = f(x) g(t) \)

\[-\frac{\hbar^2}{2m} \frac{d^2 f}{dx^2} f(x) = -\hbar \frac{dg}{dt} f(x) = -E^2 f(x)\]

\[\frac{d^2 f}{dx^2} = -\frac{2mE}{\hbar^2} f(x) \quad \frac{dg}{dt} = -\frac{iE}{\hbar} g\]

Define \( E = \frac{\hbar^2 k^2}{2m} \)

\( g(t) = e^{-\frac{iE t}{\hbar}} \)

\[\frac{d^2 f}{dx^2} = -k^2 f\]

\( f(x) = e^{ikx} \) - used sine/cosine in a box but complex exponential (plane wave) simpler here.
\[ \psi(x,t) = \int_{-\infty}^{\infty} c(k) e^{-i k x} \frac{e^{-i k^2 \Delta^2 t/2m}}{\sqrt{2\pi}} dk \]

\[ \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \]

\[ \int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{\pi}{4a^2} \sqrt{\pi} a \]

Linear combination

\[ n = \frac{1}{\Delta t} \]

\[ \psi(x,t) = n e^{-\frac{x^2}{2\Delta^2}} = \int_{-\infty}^{\infty} c(k) e^{ikx} dk \]

\[ \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\Delta^2}} dx = \sqrt{2\pi \Delta^2} \]

\[ c(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\Delta^2}} e^{-ikx} dx \]

Complete the square

\[ -\frac{x^2}{2\Delta^2} - ikx = -\frac{1}{2\Delta^2} (x + ik\Delta^2)^2 \]

\[ = -\frac{1}{2\Delta^2} (x + ik\Delta^2)^2 + \frac{1}{2\Delta^2} (ik\Delta^2)^2 \]

\[ = -\frac{1}{2\Delta^2} (x + ik\Delta^2)^2 - \frac{k^2 \Delta^4}{2} \]

Wick's theorem in quantum Field Theory

Amazingly true even if

\[ c(k) = \frac{1}{\sqrt{2\pi \Delta^2}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\Delta^2}} e^{-\frac{k^2 \Delta^2}{2} (x+ik\Delta^2)^2} dx \]

\[ \int_{-\infty}^{\infty} e^{a(x-x_0)^2} dx = \sqrt{\frac{\pi}{a}} \]

\[ x_{0 \text{ real or complex}} = \sqrt{\frac{\pi}{a}} \]

Still \( \sqrt{2\pi \Delta^2} \) even though \( i k \Delta^2 \)

in argument of exponential

\[ e(k) = \frac{n \Delta}{\sqrt{2\pi} \Delta} e^{-\frac{k^2 \Delta^2}{2}} \]
Recall $c(k)$ are coefficients of $\hat{A}$ in the new "Fourier basis": we can check normalization

$$\int_{-\infty}^{\infty} |c(k)|^2 \, dk = \eta^2 \Delta^2 \sqrt{\frac{\pi}{\Delta^2}}$$

$$\eta^2 = \frac{1}{\sqrt{\pi \Delta^2}} = \frac{1}{\Delta} \frac{1}{\sqrt{\pi}} \Delta^2 \sqrt{\frac{\pi}{\Delta^2}} = 1$$

Just as $p(x) = |A(x)|^2$ is probability to find a particle at location $x$

$$\tilde{p}(k) = |c(k)|^2$$

is probability to find a particle with momentum $\hbar k$

$$p(x) \sim e^{-x^2/\Delta^2}$$

$$\tilde{p}(k) \sim e^{-k^2 \Delta^2}$$

Q: What does this remind you of?
Uncertainty principle!

If $p(x)$ is made narrow by making $A$ small,

$p'(k)$ get wide ($\frac{1}{\lambda}$)

cannot know both $x$ and $k$ to arbitrary accuracy simultaneously.

\[
\left\{ \text{width of } p \right\} \left\{ \text{width of } k \right\} \sim 1
\]

\[
\frac{\text{width of } x}{\text{width of } k} \sim \frac{\hbar}{\lambda}
\]

$p=\pi k$

Uncertainty principle $\leftrightarrow$ Fourier integral.

It is quite messy algebraically to get $\psi(x,t)$

\[
\psi(x,t) = \int dk \, c(k) \, e^{-i\frac{k^2}{2\hbar}t - i k x} \frac{A}{\sqrt{2\pi}} e^{-k^2\lambda^2/2}
\]

Complete Square etc.
Can anyone guess what happens?

Width of wave packet spreads with time,

Stays centered at origin

An interesting calculation:

\[ \psi(x, t) = a_1 e^{-\frac{x^2}{2} \Delta^2} e^{i k_0 x} \]

Q: What happens?

A: Wave function moves and spreads
Uncertainty principle

$$\psi(x) = \eta \ e^{-x^2/2\Delta^2} \quad c(k) = \eta \Delta \ e^{-k^2\Delta^2/2}$$

$$\Delta x^2 = \langle x^2 \rangle - \langle x \rangle^2$$

(\phi \ by \ symmetry)

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 |\psi(x)|^2 \, dx = \int_{-\infty}^{\infty} \eta^2 x^2 e^{-x^2/2\Delta^2} \, dx$$

$$= \eta^2 \sqrt{\pi} \Delta^2 \frac{\Delta^2}{2} = \frac{1}{\Delta \sqrt{\pi}} \sqrt{\pi} \Delta^2 \Delta^2 = \frac{\Delta^2}{2}$$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} |c(k)|^2 (\hbar k)^2 \, dk$$

$$= \hbar^2 \int_{-\infty}^{\infty} \eta^2 \Delta^2 k^2 e^{-k^2\Delta^2} \, dk$$

$$= \eta^2 \Delta^2 \hbar^2 \sqrt{\frac{\pi}{\Delta^2}} \frac{1}{\Delta^2} = \frac{1}{\Delta \sqrt{\pi}} \Delta^2 \hbar^2 \sqrt{\frac{\pi}{\Delta^2}} \frac{1}{\Delta^2} \Delta^2$$

$$= \hbar^2 \frac{\Delta \eta^2}{\Delta^2}$$

$$\Delta x \Delta p = \frac{\Delta \hbar}{\sqrt{2} \sqrt{2\Delta}} = \frac{\hbar}{2} \text{ smallest possible value!}$$
Solving the Diffusion Eqn

\[ \frac{df(x,t)}{dt} = D \frac{\partial^2 f(x,t)}{\partial x^2} \]

Aka imaginary: \[ it \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x) \psi(x,t) \]

Schrodinger Eqn

\[ \psi(x,t) = f(x)g(t) \]

\[ f(x)g'(t) = D f''(x)g(t) \]

\[ \frac{f''(x)}{f(x)} = \frac{1}{D} \frac{g'(t)}{g(t)} = -k^2 \]

\[ f(x) = e^{ikx} \quad g(t) = e^{-Dk^2t} \]

\[ \phi(x,t) = \int dk \ c(k) e^{ikx} e^{-Dk^2t} \]

\[ \phi(x,0) = \int dk \ c(k) e^{ikx} \]

\[ c(k) = \int dx \ \frac{\psi(x,0)}{e^{ikx}} e^{-ikx} \]

Eq: \[ \phi(x,0) = \delta(x) \quad c(k) = \frac{1}{2\pi} \]

(like Schrodinger Eqn: uncertainty

perfectly localized in real span

completely spread out in K space)
\[ \psi(x,t) = \int \frac{dk}{2\pi} e^{\frac{ikx}{2\pi}} e^{-Dk^2t} \]

\[ = \frac{1}{2\pi} \int dk e^{-Dt(k^2 + \frac{ix}{Dt}z)^2 +Dt(\frac{ix}{2Dt})^2} \]

\[ = \frac{1}{2\pi} \sqrt{\frac{\pi}{Dt}} e^{-\frac{x^2}{4Dt}} \]

\[ = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} \]

Normalization: \[ \int_{-\infty}^{\infty} \psi(x,t) dx = \frac{1}{\sqrt{4\pi Dt}} \sqrt{\frac{\pi}{1/4Dt}} = 1 \]

\[ \langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \psi(x,t) dx = \frac{1}{\sqrt{4\pi Dt}} \frac{1}{\sqrt{4\pi Dt}} \frac{1}{2} \sqrt{\frac{1}{4Dt}} \]

\[ \langle x^2 \rangle = 2Dt \]

\[ \frac{\langle x^2 \rangle}{t} \sim t^{1/2} \quad \text{Diffusion} \]