

Probability, physics, and the coin toss

L. Mahadevan and Ee Hou Yong

When you flip a coin to decide an issue, you assume that the coin will not land on its side and, perhaps less consciously, that the coin is flipped end over end. What happens if those assumptions are relaxed?

L. Mahadevan (lm@seas.harvard.edu) is a professor of applied mathematics, biology, and physics at Harvard University in Cambridge, Massachusetts. **Ee Hou Yong** (ehyong@fas.harvard.edu) is a graduate student of physics at Harvard.

Why is the outcome of a coin toss random? That is, why is the probability of heads $1/2$ for a fair coin? Since the coin toss is a physical phenomenon governed by Newtonian mechanics, the question requires one to link probability and physics via a mathematical and statistical description of the coin's motion. However, that is not typically how one approaches the question. An empirical approach based on repeated experiments might suggest that the result is approximately correct. Another route is based on symmetry; since a coin of zero thickness can land on either of two equivalent faces, the probabilities for each must be the same. But such is clearly not always true. For example, a coin that does not flip even once will end up the same way it started. And even if it flips, it might not do so frequently; instead, it could wobble like a Frisbee and thus still be biased to land with its starting side up.

Randomness defined

The considerations noted above raise a fundamental issue in probability, termed Bertrand's paradox. The idea is that in a random process, probabilities are ill-defined unless one specifies the nature of the process that leads to the random variable. To illustrate the principle in the context of a coin toss, we pose the following question: How thick should a coin be to have a $1/3$ chance of landing on edge? John von Neumann is said to have solved the problem instantly on hearing of it, giving 0.354 for the aspect ratio (thickness divided by diameter)—a three-decimal approximation of $1/(2\sqrt{2})$.

But how did he answer the question? Presumably, he assumed that all possible orientations of the coin are equally likely. Then the question boils down to asking what the thickness of the coin should be so that the areas of its sides and the faces are equal when projected onto the circumscribing sphere that characterizes the possible orientations; figure 1a shows the geometry. But von Neumann's mathematically plausible interpretation is impossible for a real tossed coin, which must conserve angular momentum and thus cannot explore all possible orientations. For example, the possible orientations of a coin spun end over end about a diameter are limited to a circle, not a sphere. Consequently, the condition of fairness leads to a different answer, as shown in figure 1b. Clearly, the process underlying the generation of a random variable matters.

Get physical

Endowing probability with an underlying physical basis is a natural way to build in a mechanism for randomness. The approach has antecedents going back to Pierre Simon Laplace and more directly to Henri Poincaré, who analyzed the game of roulette. Poincaré addressed the question of how small variations in initial conditions and the physics of collisions determine the game's probabilistic outcomes. Later Eberhard Hopf showed how the nearly constant observed frequencies of an event—frequencies consistent with statistical inference—can naturally arise from the underlying physical processes.

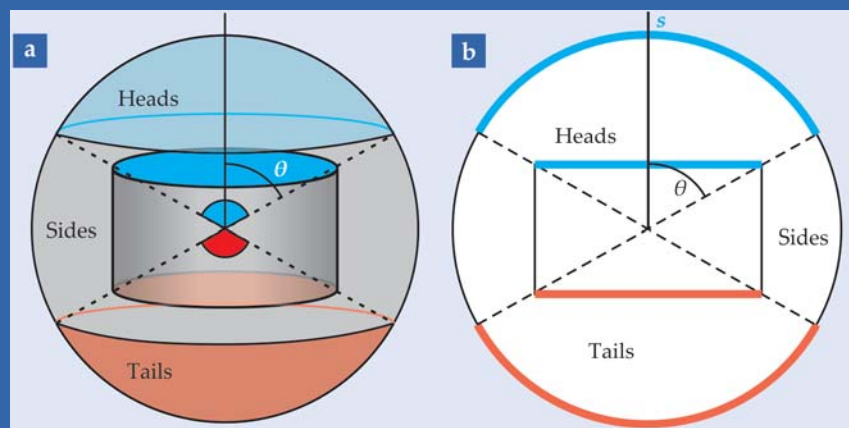


Figure 1. Bertrand's paradox and the toss of a thick coin. The question "What is the aspect ratio ξ —the ratio of height to diameter—for a fair, thick coin?" can lead to different answers depending on the underlying assumptions associated with the mechanism that leads to the generation of possible outcomes. **(a)** If the coin can assume all possible orientations in three-dimensional space with equal probability, the probability of heads is $\Omega_s/4\pi$, the ratio of the solid angle Ω_s subtended by the head face of the coin to the total solid angle of the circumscribing sphere. For a fair coin, $\Omega_s = 4\pi/3 = 2\pi(1 - \cos \theta)$. Given that $\cos \theta = \xi/(1 + \xi^2)^{1/2}$, $\xi = 1/(2\sqrt{2})$.

(b) For the dynamical case of a coin flipped end over end the probability of heads changes since the geometry of orientation space changes. Here, the probability of heads is $s/2\pi$, the ratio of the arc length s subtended by the heads face and the circumference of the circle. For a fair three-sided coin, $s = 2\pi/3$ and so $\xi = 1/\sqrt{3}$.

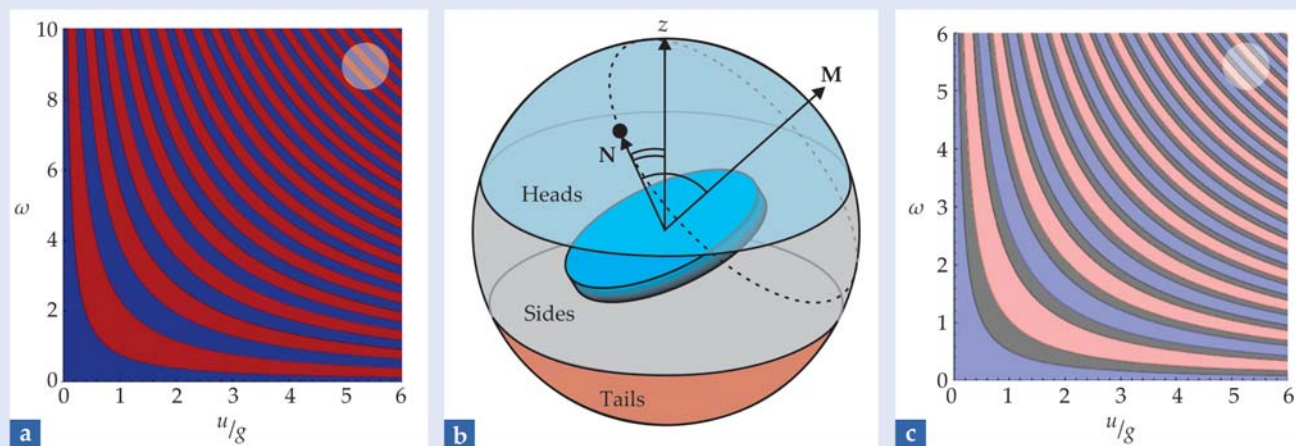


Figure 2. Geometry, dynamics, and probability in a coin toss. (a) In 1986 Joseph Keller analyzed the end-over-end spinning of a zero-thickness coin launched heads up with spin ω and vertical speed u that lands without bouncing. The phase space of initial conditions for ω and u (scaled by the gravitational acceleration g) is tiled into heads (blue) and tails (red). The hyperbolas bounding the tiles satisfy the equations $\omega = (2n \pm 1/2) \cdot \pi g / 2u$, with $n = 0, 1, 2, \dots$, which follow from the solution of the equations of motion. As ω and u/g become large, any disk representing a probability distribution of initial conditions is very finely tiled by heads and tails regions that occupy a fixed, equal fraction of the disk. Thus vigorously spinning coins show no bias, and the probabilities for heads and tails become equal. (b) For a spinning, precessing coin whose heads face has a normal vector $\mathbf{N}$, conservation of angular momentum $\mathbf{M}$ dictates that $\mathbf{N}$ precesses about $\mathbf{M}$, sweeping out a circle on the circumscribing sphere. Only when $\mathbf{N}$ and $\mathbf{M}$ are perpendicular can the coin be fair as discussed in figure 1b. (c) For a fair thick coin, the hyperbolas analogous to those given in panel a separate the phase space into regions of heads (purple), sides (gray), and tails (pink). As ω and u/g become large, any disk representing a probability distribution of initial conditions is tiled finely and equally, now by regions associated with heads, tails, and sides.

Exactly how physics and probability come together in the coin-toss problem was analyzed by Joseph Keller, who studied a coin of zero thickness that spins end over end without air resistance and lands without bouncing. Keller proved mathematically that the idealized coin becomes fair only in the limit of infinite vertical and angular velocity. His elegant argument is summarized in the caption for figure 2a.

Get real, get thick

Real coins spin in three dimensions and have finite thickness. Building on Keller's work, Persi Diaconis, Susan Holmes, and Richard Montgomery analyzed the three-dimensional dynamics of a spinning, tumbling rigid body as applied to coins with zero thickness but arbitrary angular momentum $\mathbf{M}$. Conservation of angular momentum implies that the vector normal to the heads face of the coin precesses, and allowed the three researchers to derive simple explicit formulas for the probability distribution of heads and tails in the limit of large spin and speed. They predicted and experimentally verified that a vigorously flipped coin is biased by its initial state and is truly fair only when it spins end over end—in other words, only when it follows the Keller flip.

Adding a finite thickness to the coin allowed us to revisit the question of a fair three-sided coin in a dynamical setting. As expected, the normal vector precesses around $\mathbf{M}$, as illustrated in figure 2b. A simple analysis essentially following that of Diaconis and company enabled us to calculate the probability of heads as a function of the aspect ratio of the coin and the angle between $\mathbf{M}$ and the initial orientation of the normal vector. We found that vigorously tossed coins are biased to come up as they have started unless flipped end over end, and we verified that a fair three-sided coin should have an aspect ratio of $1/\sqrt{3}$.

Our experiments with thick coins that are spun vigorously and allowed to land inelastically confirm our prediction for the geometry of a fair coin. The phase space of initial conditions in this case is decomposed into regions shown in

figure 2c and tiled by the three possibilities of heads, tails, and sides; the sides regions occur twice as often as those for heads and tails, but they have only half the area.

Clearly, one could add more physical realism and fun to a description of the coin toss by accounting for fluid resistance, bouncing, rolling, and so forth. For example, the effect of fluid resistance is relevant for the parlor game of dropping a coin toward a target at the bottom of a water-filled jar, and it increases the complexity of the problem enormously. The rolling of a polygonal object such as a pencil provides a simple model for the dynamics of bouncing and has interesting connections to the physics of footfall in robots. Many riches remain to be mined by the study of coin tosses and other “simple” mechanical games of chance. And there may be literal riches too. Tom Stoppard's play *Rosencrantz and Guildenstern Are Dead* gets off to an incredible start when Rosencrantz wins 92 bets in a row by wagering on heads. But learning how to toss a coin so that it looks like it is flipping even as it only wobbles can make the feat a reality.

Additional resources

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