Midterm Exam

[1.] Consider the vector potential \( \mathbf{A}(r) = A_0 e^{-(x^2+y^2)/a^2} \hat{z} \), where \( A_0 \) and \( a \) are constants.
(a) Find and sketch the corresponding magnetic field. How is it similar/different from the field due to a long straight wire?
(b) Can this be a magnetostatic field? If yes, find the current distribution that would give rise to it, and if not, explain why not.
(c) Is the vector potential given in the Coulomb gauge?

\[
\begin{align*}
\mathbf{B} &= \nabla \times \mathbf{A} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
0 & 0 & A_0 e^{-(x^2+y^2)/a^2}
\end{vmatrix} \\
&= \frac{2A_0}{a^2} \left( -y \hat{x} + x \hat{y} \right) e^{-(x^2+y^2)/a^2}
\end{align*}
\]

The vector structure is the same as a long straight wire. The decay is much faster \( e^{-(x^2+y^2)/a^2} \) vs. \( \sqrt{x^2+y^2} \).

b) We need to check if \( \nabla \cdot \mathbf{J} = \partial \rho / \partial t = 0 \)
we get \( \mathbf{J} \) from \( \frac{1}{\sqrt{4\pi}} \mathbf{E} \times \mathbf{B} \)

\[
\nabla \times \mathbf{B} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
0 & 0 & 2A_0/a^2
\end{vmatrix}
\]

\[
= \frac{2A_0}{a^2} \hat{z} \left( 1 - \frac{2x^2}{a^2} + 1 - \frac{2y^2}{a^2} \right) e^{-(x^2+y^2)/a^2}
\]

\[
\frac{4\pi}{c} \mathbf{J} = \frac{4\pi}{c} \hat{z} \left( 1 - \frac{x^2+y^2}{a^2} \right) e^{-(x^2+y^2)/a^2}
\]

Clearly \( \nabla \cdot \mathbf{J} = 0 \) because \( J_x = J_y = 0 \) and \( J_z \) has no \( z \) dependence.

\[\text{c) For the same reason that } \nabla \cdot \mathbf{J} = 0 \text{ we also have } \nabla \cdot \mathbf{A} = 0 : \]

\[
A_x = A_y = 0 \quad \text{and} \quad \frac{\partial A_z}{\partial z} = 0, \quad \text{so, yes, we are in Coulomb gauge.}
\]
[2.] In your homework, you solved for the Greens function of the diffusion equation,

$$\nabla^2 \vec{A} = \mu \sigma \frac{\partial}{\partial t} \vec{A}$$  \hspace{1cm} (1)

Show that the vector potential $\vec{A}$ does indeed obey the diffusion equation in a conducting medium where the current density $\vec{J} = \sigma \vec{E}$. You will need to use Maxwell’s equations and the usual relations between the fields $\vec{E}$, $\vec{B}$ and the potentials $\Phi$, $A$. You may assume there is no free charge so that $\Phi = 0$ and work in the Coulomb gauge $\nabla \cdot \vec{A} = 0$.

If a current is flowing in a conductor, producing a vector potential $\vec{A}$, and then the current is suddenly turned off, describe what Eq. 1 tells you qualitatively about how $\vec{A}$ evolves. Assume the conductor fills all space.

We have

$$\nabla \times \vec{B} = \mu \vec{J} = \mu \sigma \vec{E}$$

in a conductor.

Also

$$\vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t} = -\frac{\partial \vec{A}}{\partial t}$$

in the absence of free charges.

Finally

$$\vec{B} = \nabla \times \vec{A}$$

leading to

$$\nabla \times (\nabla \times \vec{A}) = -\mu \sigma \frac{\partial \vec{A}}{\partial t}$$

$$\nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$\downarrow$$

So we get Diffusion Eqn

$$\nabla^2 \vec{A} = \mu \sigma \frac{\partial \vec{A}}{\partial t}$$
The Greens function for the one dimensional diffusion equation obeys

\[ (D \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}) G(x, t) = \delta(x) \delta(t). \] (2)

Write

\[ G(x, t) = \int dk \int d\omega \ e^{i(kx - \omega t)} G(k, \omega) \] (3)

and find a formula for $G(k, \omega)$. Insert this into Eq. 3 and do the $\omega$ integration. How does causality appear from the mathematics? If you have time, do the $k$ integration as well, or if not, state the general procedure for doing it.

Plugging (3) into (2) and identifying the coefficients $G(k, \omega)$ on each side

\[-DK^2 + i\omega \] $G(k, \omega)$ $\frac{1}{(2\pi)^2}$

Here we used $S(x) = \frac{1}{2\pi} \int dx \ e^{ikx}$

and similarly for $\delta(t)$

\[ G(k, \omega) = \frac{1}{(2\pi)^2} \frac{1}{i\omega - DK^2} \]

Inserting this into (3) to get $G(x, t)$

\[ G(x, t) = \int \frac{dk \ int \ d\omega \ e^{i(kx - \omega t)}}{(2\pi)^2} \frac{1}{i\omega - DK^2} = \text{pole at } \omega = -iDK^2 \]

The $\omega$ integration uses $e^{-i\omega t} = e^{-i\omega t}$ for $t < 0$ we need to close contour with $\omega i > 0$ to get $e^{i\omega t} \to 0$

We do not enclose the pole and $G(x, t) = 0$.

For $t > 0$ we need to close contour with $\omega i < 0$ (lower half plane) and we do pick up the pole. We get $2\pi i \times \text{Residue}$

\[ G(x, t) = -i \int \frac{dk}{2\pi} \ e^{i(kx - \omega t)} \ e^{-i(-i DK^2) t} \] \( \theta(t) = -i \int \frac{dk}{2\pi} e^{-DK^2 t + ikx} \theta(t) \)

The $k$ integral is done via completing the squares

$-DK^2 t + ikx = -D t (k - \frac{i x}{2D t})^2 - x^2/4Dt$

\[ G(x, t) = \frac{i}{2\pi} e^{-x^2/4Dt} \theta(t) \sqrt{\frac{\pi}{D t}} \]
We expect \( G(x,t) \) is normalized to 1 because
\( G(x,t) \) is supposed to represent the solution to the diffusion equation with a \( \delta(x) \) source, and the diffusion equation preserves the particle number. Let's check

\[
\int_{-\infty}^{\infty} dx \frac{1}{\sqrt{4\pi D t}} e^{-x^2/4D t} = \frac{1}{2} \sqrt{\frac{\pi}{D t}} \int_{0}^{\infty} 4\pi D t = 1 \]

(\#) What this means is that if \( G(x,t) \) is a solution of the diffusion equation

\[
\frac{\partial}{\partial t} \int_{-\infty}^{\infty} G(x,t) dx = \int_{-\infty}^{\infty} \frac{D}{\partial x^2} G(x,t) dx
\]

2 integrate by parts

\[
= D \frac{\partial}{\partial x} \left[ -\int_{-\infty}^{\infty} \phi dx \right] - \int_{-\infty}^{\infty} D \frac{\partial^2}{\partial x^2} \phi dx
\]

\[
\phi \quad \text{assumed} \quad \phi \text{ and } \frac{\partial \phi}{\partial x} \text{ vanish at } \pm \infty
\]

so

\[
\frac{\partial}{\partial t} \int_{-\infty}^{\infty} G(x,t) dx = 0
\]