\[ R^T R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \]

\[ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \]

general proof

\[ V' = MV \]

\[ (V')^T = V^T M^T \quad \text{remember to interchage when transposing!} \]

\[ (V')^T V' = (V^T M^T)(M V) \]

\[ = V^T (M^T M)V \quad \text{matrix multiplication is associative} \]

\[ = V^T I V \]

\[ = V^T V \]
Generalization of transpose to complex matrices

is "Hermitean conjugate"

\[(A^+)^* = A \] transpose and complex conjugate

If \( A \) is real \( A^+ = A^T \)

Real Matrices

Complex Matrices

\[ AA^T = I \]

Orthogonal

\[ AA^+ = I \]

Unitary

\[ A = A^T \]

Symmetric

\[ A = A^+ \]

Hermitian

Unitary matrices preserve lengths of vectors which have complex components,

\[ \sum v_n^2 = 1 \quad \text{real} \]

\[ \sum |v_n|^2 = 1 \quad \text{complex} \]

\[ V^T V = 1 \quad \text{real} \]

\[ V^+ V = 1 \quad \text{complex} \]

\[ V' = AV \]

\[ (V')^+ = V^+ A^+ \]

\[ (V')^+ V' = V^+ A^+ A V \]

\[ \text{I} \]
\[ \det A B = \det A \det B \quad \text{(not easy to prove!)} \]

\[ \det M^T = \det M \]

For a symmetric matrix:

\[ \det M M^T = \det I = 1 \]

\[ \Rightarrow \det M \det M^T = (\det M)^2 \]

\[ \Rightarrow \det M = \pm 1 \]

Rotation matrix:

\[ \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad \det M = 1 \]

Reflection in x-axis:

\[ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \det M = -1 \]
Matrices in QM

Vector $V$ → components $V_n$
  $\uparrow$ discrete label
  $\left\{ \text{usually finite} \right\}$

Wave function $|\Psi\rangle$ → components $\Psi(x)$
  $\left\{ \text{usual notation in QM} \right\}$
  Instead of $\Psi$
  $\rightarrow$ continuous label
  $\left\{ \text{Matrices in QM} \right\}$
  $\rightarrow$ infinite #
  $\left\{ \text{are infinite dimensional!} \right\}$

\[ |\Psi(x)|^2 = \text{probability to find qm particle at } x \]
\[ \sum |V_n|^2 = 1 \leftarrow \text{vector is normalized} \]
\[ \int dx |\Psi(x)|^2 = 1 \leftarrow \text{sum of probabilities is 1} \]

So normalization of vector in QM is required for interpretation of $|\Psi(x)|^2$ as probability

$|\Psi(t)\rangle$

$\Psi(x,t)$ really because wave function depends on time

CM:
\[ X(t+dt) = x(t) + v(t)dt \]
\[ v(t+dt) = v(t) + f(x,v)/m dt \]

QM:
\[ |\Psi(t+dt)\rangle \leftarrow |\Psi(t)\rangle \]

$\uparrow$ matrix gives vector $|\Psi(t+dt)\rangle$ from vector $|\Psi(t)\rangle$
Nature of Matrix

If $|\psi(t)\rangle$ has length 1

we want $|\psi(t+\Delta t)\rangle$ to have length 1 also

$|\psi(t+\Delta t)\rangle = U|\psi(t)\rangle$

we will follow up move on this later but for now, connect to your homework

$e^{-i\sigma_x}$

$e^{-i\sigma_y}$

$e^{-i\sigma_z}$

You are computing these.

They are all unitary.

you can verify this

It turns out these matrices are some of the ones that evolve spin $\frac{1}{2}$ wave function.

Spin $\frac{1}{2}$ is a very nice QM problem because unlike $\psi(x)$ in infinite dim spin $\frac{1}{2}$ is 2 dim.
We saw (when we were young) matrices arising in linear algebra problems. They also arise in QM (as we just started to see) and in CM, let's look at CM first.

We did one mass on a spring. Now 2:

\[ m_1 \ddot{x}_1 = -k(x_1 - x_2) \quad \text{Newton's Third Law.} \]
\[ m_2 \ddot{x}_2 = -k(x_2 - x_1) \]

One (non matrix) approach

Add \[ m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = 0 \]

\[ m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = C \quad \text{what is this?} \]

\[ \frac{d}{dt} (m_1 \dot{x}_1 + m_2 \dot{x}_2) = \frac{d}{dt} (m_1 v_1 + m_2 v_2) = 0 \]

Momentum conservation. \( \Leftrightarrow \) no "external forces!"
Set \( M_1 = M_2 = m \) for simplicity.

Subtract:

\[
\begin{align*}
mx_1^\circ &= -k(x_1 - x_2) \\
mx_2^\circ &= -k(x_2 - x_1)
\end{align*}
\]

\[
m(x_1^\circ - x_2^\circ) = -2k(x_1 - x_2)
\]

\[
\frac{d^2}{dt^2}(x_1 - x_2) = -\frac{2k}{m}(x_1 - x_2)
\]

"relative coordinate" \( x = x_1 - x_2 \)

\[
x^\circ = -\frac{2k}{m}x
\]

Oscillatory with \( \omega^2 = \frac{2k}{m} \)

"center of mass coordinate" \( \bar{x} = \frac{x_1 + x_2}{2} \)

\[
\ddot{x} = 0 \Rightarrow \bar{x} = C + D t
\]

\[
x(t) = x_1(t) - x_2(t) = A \cos(\omega t) + B \sin(\omega t)
\]

\[
\bar{x}(t) = x_1(t) + x_2(t) = C + D t
\]

What determines \( A, B, C, D \)? \( x_1(0), x_2(0), \dot{x}_1(0), \dot{x}_2(0) \)

The initial conditions

\[
mx = -\omega^2 x \quad \text{has oscillating solutions, cos} \omega t, \sin \omega t
\]

except when \( \omega = 0 \):

\[
\ddot{x} = 0 \Rightarrow x = C + D t
\]
\[ x_1(0) - x_2(0) = A \quad x_1(0) + x_2(0) = C \]
\[ \dot{x}_1(0) - \dot{x}_2(0) = \omega B \quad \dot{x}_1(0) + \dot{x}_2(0) = D \]

Then, of course

\[ x_1(t) = \frac{1}{2} (x_1(t) + \bar{x}(t)) \]
\[ x_2(t) = \frac{1}{2} (-x_1(t) + \bar{x}(t)) \]

How to generalize so many masses?

This is best done with matrices.

Preview of:

"Normal modes"
Let's redo 2 mass case

Guess solution \( x_1 = V_1 e^{i\omega t} \) \( \omega \) same \( \omega \)!

\[ x_2 = V_2 e^{i\omega t} \]

\[ \text{will it work?} \]

It's just a guess of ours \( \text{Not obvious} \)

\[ -m \omega^2 V_1 e^{i\omega t} = -k (V_1 - V_2) e^{i\omega t} \]

\[ -m \omega^2 V_2 e^{i\omega t} = -k (V_1 - V_2) e^{i\omega t} \]

\[ \left( \begin{array}{cc} k - m \omega^2 & -k \\ -k & k - m \omega^2 \end{array} \right) \left( \begin{array}{c} V_1 \\ V_2 \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \]

\[ \det \text{ Must vanish} \]

\[ m (k - m \omega^2)^2 - k^2 = 0 \]

\[ k - m \omega^2 = \pm k \]

\[ m \omega^2 = 0, 2k \]

\[ \omega^2 = 0 \quad \omega^2 = 2k/m \]

\[ \text{Does this remind anyone of anything? EIGENVALUES} \]