be calculated reliably. The problem may be addressed using full grid calculations or other expansion basis may be considered, however, for finite density such a basis should include non-analytical functions.

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PHASE TRANSITIONS IN NON-HERMITEAN MATRIX MODELS AND THE "SINGLE RING" THEOREM

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Recently, an analytic method was developed to study in the large N limit non-hermitean random matrices that are drawn from a large class of circularly symmetric non-Gaussian probability distributions, thus extending the existing Gaussian non-hermitean literature. One obtains an explicit algebraic equation for the integrated density of eigenvalues from which the Green's function and averaged density of eigenvalues could be calculated in a simple manner. Thus, that formalism may be thought of as the non-hermitean analog of the method due to Brèzin, Itzykson, Parisi and Zuber for analyzing hermitean non-Gaussian random matrices. A somewhat surprising result is the so called "Single Ring" theorem, namely, that the domain of the eigenvalue distribution in the complex plane is either a disk or an annulus. In this talk we provide a non-technical review of these results. We also report on new sum-rules for the radii of the eigenvalue distiobution and for the value of the eigenvalue density at the edges of the eigenvalue distribution of the non-hermitean matrix in terms of moments of the eigenvalue distribution of the associated hermitean matrix. Our analytical predictions are supported by numerical results.

1 Introduction

This talk[a] is intended to give a non-technical review of our recent results on random non-hermitean matrices. A detailed exposition with comprehensive bibliography will appear in the near future[1].

[a] Talk delivered by Joshua Feinberg

There has been considerable interest in random non-hermitean matrices in recent years. For some recent reviews see[2].

One difficulty is that the eigenvalues of non-hermitean matrices invade the complex plane, and consequently, various methods developed over the years to deal with random hermitean matrices are no longer applicable, as these methods typically all involve exploiting the powerful constraints of analytic function theory. (See in particular the paper by Brézin, Itzykson, Parisi, and Zuber [3].) Some time ago, two of us proposed [4] a "method of hermitization", whereby a problem involving random non-hermitean matrices can be reduced to a problem involving random hermitean matrices, to which various standard methods (such as the diagrammatic method[5], or the "renormalization group" method[6,7,8,9]) can be applied. An idea similar to the "method of hermitization" was expressed independently in [10].

To our knowledge, the literature on random non-hermitean matrices has focussed exclusively on Gaussian randomness. For instance, it has been known for over thirty years, from the work of Ginibre[12], that for the Gaussian probability distribution $P(\phi) = (1/Z)\exp\,(-N\mathrm{tr}\phi^\dagger\phi)$ (here, as in the rest of this talk, ϕ denotes an $N \times N$ complex random matrix with the limit $N \to \infty$ understood), the density of eigenvalues of ϕ is uniformly distributed over a disk of radius 1 in the complex plane.

Analytic determination of the density of eigenvalues of a non-Gaussian probability distribution of the form

$$P(\phi) = \frac{1}{Z} e^{-N\mathrm{tr}V(\phi^\dagger\phi)}\,, \tag{1}$$

where V is an arbitrary polynomial of its argument, was given for the first time in [11]. Based on the method of hermitization, it was shown in [11] that by a simple trick, the desired density of eigenvalues could be obtained with a minimal amount of work, by judiciously exploiting the existing literature on random hermitean matrices.

Due to the symmetry of $P(\phi)$ under the transformation $\phi \to e^{i\alpha}\phi$, the density of eigenvalues is obviously rotational invariant. It was shown in [11] that the class of probability distributions of the form (1) exhibits a universal behavior in the sense that whatever the polynomial V was, the shape of the eigenvalue distribution in the complex plane was always either a disk or an annulus. This result was referred to in [11] as the "Single Ring Theorem".

In a certain sense, the formalism developed in [11] may be thought of as the analog of the work of Brézin et al. for random hermitean matrices [3]; they showed how the density of eigenvalues of hermitean matrices φ taken from the probability distribution $P(\varphi) = (1/Z)\exp[-N\mathrm{tr}V(\varphi)]$ with V an arbitrary polynomial can be determined, and not just for the Gaussian case studied by Wigner and others[13], in which $V = (1/2)\mathrm{tr}\varphi^2$. An important simplifying feature of the analysis in [3] is that $P(\varphi)$ depends only on the eigenvalues of φ, and not on the unitary matrix that diagonalizes it. In contrast, the probability distribution (1) for non-hermitean matrices depends explicitly on the $GL(N)$ matrix S used to diagonalize $\phi = S^{-1}\Lambda S$, and S does not decouple. Remarkably however, for the Gaussian $P(\phi)$, Ginibre [12] managed to integrate over S explicitly and derived an explicit expression for the probability distribution of the eigenvalues of ϕ. Unfortunately, it is not clear how to integrate over S and derive the expression for the eigenvalue probability distribution for non-Gaussian distributions of the form (1). In [11] this difficulty was circumvented by using the method of hermitization.

2 The Method of Hermitization and Non-Gaussian Ensembles

Here we very briefly summarize the "method of hermitization" [4,11] in the form of an algorithm, followed by a general discussion of the phase structure of the eigenvalue distribution, but first introduce some notations and definitions.

The averaged density of eigenvalues

$$\rho(x,y) = \langle \frac{1}{N}\sum_i \delta(x - \mathrm{Re}\,\lambda_i)\,\delta(y - \mathrm{Im}\,\lambda_i)\rangle \tag{2}$$

of the non-hermitean matrix ϕ, may be determined from the the Green's function associated with ϕ, namely

$$G(z,z^*) = \langle \frac{1}{N}\mathrm{tr}\frac{1}{z-\phi}\rangle = \int d^2x'\,\frac{\rho(x',y')}{z-z'}\,, \tag{3}$$

in terms of which[b]

$$\rho(x,y) = \frac{1}{\pi}\partial^*\,G(z,z^*)\,. \tag{4}$$

The probability distributions (1) are invariant under $\phi \to e^{i\alpha}\phi$, rendering

$$\rho(x,y) \equiv \rho(r)/2\pi \tag{5}$$

circularly invariant. Rotational invariance thus leads to a simpler form of the defining formula (3) for $G(z,z^*)$ which reads

$$\gamma(r) \equiv zG(z,z^*) = \int_0^r r'dr'\,\rho(r')\,, \tag{6}$$

[b] We use the following notational conventions: for $z = x + iy$ we define $\partial \equiv \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ so that $\partial z = 1$. Similarly, we define $\partial^* \equiv \frac{\partial}{\partial z^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, so that $\partial^* z^* = 1$ and also $\partial^*(1/z) = \pi\delta^{(2)}(z)$. Finally, we denote $|z| = r$.

whence

$$\rho(r) = \frac{1}{r}\frac{d\gamma}{dr}\,. \tag{7}$$

Clearly, the quantity $\gamma(r)$, which can be thought of as the integrated eigenvalue density, is a positive monotonically increasing function, which satisfies the obvious "sum-rules"

$$\gamma(0) = 0 \qquad \text{and} \qquad \gamma(\infty) = 1\,. \tag{8}$$

It was shown in [11] that by applying a simple trick, the desired Green's function of a non-hermitean random matrix ϕ could be obtained with a minimal amount of work, by judiciously exploiting the existing literature on random hermitean matrices. The algorithm, according to [11], for finding the Green's function and the averaged eigenvalue density of a non-hermitean random matrix ϕ drawn from a non-Gaussian ensemble $P(\phi) = (1/Z)e^{-N\mathrm{tr}V(\phi^\dagger\phi)}$ (Eq. (1)) is as follows:

Start with the Green's function

$$F(w) = \langle \frac{1}{N}\mathrm{tr}_{(N)}\ \frac{1}{w-\phi^\dagger\phi}\rangle \equiv \int_0^\infty \frac{\tilde{\rho}(\sigma)d\sigma}{w-\sigma}\,, \tag{9}$$

where

$$\tilde{\rho}(\mu) = \frac{1}{N}\langle \mathrm{tr}_{(N)}\ \delta(\mu-\phi^\dagger\phi)\rangle \tag{10}$$

is the averaged eigenvalue density of $\phi^\dagger\phi$. Then, the desired equation for $\gamma(r) \equiv zG(z,z^*)$ is

$$\gamma\left[r^2\ F\left(\frac{\gamma\, r^2}{\gamma-1}\right) - \gamma + 1\right] = 0\,. \tag{11}$$

Thus, given F one can solve for $\gamma(r)$ using this master equation.

Of course, $F(w)$ is already known in the literature on chiral and rectangular block random hermitean matrices for the Gaussian distribution [9,15,16,17], as well as for non-Gaussian probability distributions of the form (1) with an arbitrary polynomial potential $V(\phi^\dagger\phi)$ [18,19,20]. In the latter case, for V a polynomial of degree p, the Green's function $F(w)$ is given[9,19,20] by[c]

$$F(w) = \frac{1}{2}V'(w) - P(w)\sqrt{(w-a)(w-b)}\,, \tag{12}$$

[c]Here we assume for simplicity that the eigenvalues of $\phi^\dagger\phi$ condense into a single segment $[a,b]$.

where

$$P(w) = \sum_{k=-1}^{p-2} c_k\ w^k\,. \tag{13}$$

The real constants $0 \le a < b$ and c_k are then determined completely by the requirement that $F(w) \to \frac{1}{w}$ as w tends to infinity, and by the condition that $F(w)$ has at most an integrable singularity as $w \to 0$. Thus, if $a > 0$, inevitably $c_{-1} = 0$. However, if $a = 0$, then c_{-1} will be determined by the asymptotic behavior for w large.

Eq.(11) is an algebraic equation for $\gamma(r)$ and thus may have several r dependent solutions. In constructing the actual $\gamma(r)$ one may have to match these solutions smoothly into a single function which increases monotonically from $\gamma(0) = 0$ to $\gamma(\infty) = 1$.

As a trivial example consider the Gaussian case, $V = \phi^\dagger\phi$. We have $2\sqrt{w}F(w) = \sqrt{w} - \sqrt{w-4}$, whence the roots of (11) are $\gamma = 0, 1$ and r^2. We note that $\gamma = 0$ is unphysical, $\gamma = r^2$ (*i.e.*, $G = z^*$) corresponds to Ginibre's disk[12], and $\gamma = 1$ is the solution outside the disk.

The first non-trivial example, the case $V(\phi^\dagger\phi) = 2m^2\phi^\dagger\phi + g(\phi^\dagger\phi)^2$, was studied in detail in [11]. For m^2 positive, the eigenvalue distribution was disk-like (and non uniform), generalizing Ginibre's work. In Fig. 1 we compare the analytical results of [11] for $\rho(r)$ in the disk phase of the quartic ensemble with results of Monte-Carlo measurements. The agreement is convincing. As $m^2 \equiv -\mu^2$ was made more and more negative, a phase transition at the critical value $\mu_c^2 = \sqrt{2g}$ occured, after which the disk fragmented into an annulus. Results for the annular phase of the quartic ensemble are displayed in Fig.2

A remarkable property of (11) is that it has only two r-independent solutions: $\gamma = 0$ and $\gamma = 1$ [11]. Since the actual $\gamma(r)$ increases monotonically from $\gamma(0) = 0$ to $\gamma(\infty) = 1$, we immediately conclude from this observation that there can be no more than a single void in the eigenvalue distribution. Thus, in the class of models governed by $P(\phi) = \frac{1}{Z}e^{-N\mathrm{tr}V(\phi^\dagger\phi)}$ (Eq. (1)), the shape of the eigenvalue distribution is either a disk or an annulus, whatever polynomial the potential $V(\phi^\dagger\phi)$ is. This result is the "Single Ring Theorem" of [11].

The "Single Ring Theorem" may appear counter-intuitive at first sight. Indeed, consider a potential $V(\phi^\dagger\phi)$ with several wells or minima. For deep enough wells, we expect the eigenvalues of $\phi^\dagger\phi$ to "fall into the wells". Thus, one might suppose that the eigenvalue distribution of ϕ to be bounded by a set of concentric circles of radii $0 \le r_1 < r_2 < \cdots < r_{n_{\max}}$, separating annular regions on which $\rho(r) > 0$ from voids (annuli in which $\rho(r) = 0$.)

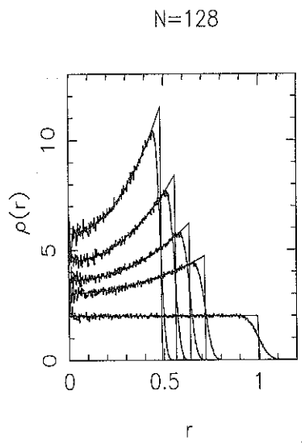

Figure 1. Comparison between Monte-Carlo measurements of the density of eigenvalues $\rho(r)$ of matrices ϕ of size 128x128, taken from the quartic ensemble $V(\phi^\dagger\phi) = 2m^2\phi^\dagger\phi + g(\phi^\dagger\phi)^2$ with $m^2 = 0.5$ (disk phase) and for $g = 0, 0.5, 1, 2, 4$ (g increases from bottom to top), compared to the analytical results of [11] (solid lines). At $g = 0$ we obtain Ginibre's Gaussian ensemble with $V = \phi^\dagger\phi$, with its unit disk of eigenvalues.

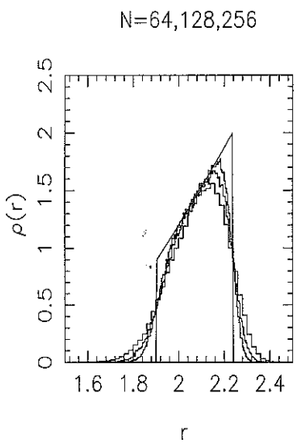

Figure 2. Comparison between Monte-Carlo measurements of the density of eigenvalues $\rho(r)$ of matrices ϕ of various size, taken from the quartic ensemble with $m^2 = -\mu^2 = -0.5$ (annluar phase) for $g = 0.1$, compared to the analytical results of [11] (solid line). Note the convergence of the numerical results to the analytical results, as N is increased.

A priori, it is natural to assume that the maximal number of such circular boundaries should grow with the degree of V, because V may then have many deep minima. Remarkably, however, according to the "Single Ring Theorem" the number of these boundaries is two at the most. To reconcile this conclusion with the a priori expectation just mentioned, note that while the eigenvalues of the hermitean matrix $\phi^\dagger\phi$ may split into several disjoint segments along the positive real axis, this does not necessarily constrains the eigenvalues of ϕ itself to condense into annuli. Indeed, the hermitean matrix $\phi^\dagger\phi$ can always be diagonalized $\phi^\dagger\phi = U^\dagger\Lambda^2 U$ by a unitary matrix U, with $\Lambda^2 = \mathrm{diag}(\lambda_1^2, \lambda_2^2, \cdots, \lambda_N^2)$, where the λ_i are all real. This implies that $\phi = V^\dagger\Lambda U$, with V a unitary matrix as well. Thus, the complex eigenvalues of ϕ are given by the roots of $\det(z - \Lambda W) = 0$, with $W = UV^\dagger$. Evidently, as W ranges over $U(N)$, the eigenvalues of ΛW could be smeared (in the sense that they would not span narrow annuli around the circles $|z| = |\lambda_i|$.)

According to the "Single Ring" theorem [11], the eigenvalue distribution of ϕ is either a disk or an annulus. The behavior of $F(w)$ as $w \sim 0$ turns out to be an indicator as to which phase of the two the system is in.

A. Disk Phase: In the disk phase $\gamma(r) \sim \rho(0)r^2/2$, with $\rho(0) > 0$, of course (see (7)). Using this simple fact, we show in [1] that in the disk phase F has the form

$$F_{disk}(w) = \frac{1}{2}V'(w) - \left(\sqrt{\frac{\rho(0)}{2b}}\, w^{-1} + c_0 + c_1\, w + \cdots + c_{p-2}\, w^{p-2}\right)\sqrt{w(w-b)}\,. \tag{14}$$

B. Annular Phase: In the annuluar phase $\gamma(r)$ must clearly vanish identically in the inner void of the annulus. Thus, (11) implies that $F(w)$ cannot have a pole at $w = 0$, and therefore from (12) we must have $c_{-1}\sqrt{ab} = 0$. Thus, the annulus must arise for $c_{-1} = 0$ (the other possible solution $a = 0, c_{-1} \neq 0$ leads to a disk configuration with $\gamma = 0$ only at $r = 0$, as we just discussed.) Thus, in the annular phase

$$F_{annulus}(w) = \frac{1}{2}V'(w) - \left(c_0 + c_1\, w + \cdots + c_{p-2}\, w^{p-2}\right)\sqrt{(w-a)(w-b)}\,. \tag{15}$$

Having determined $F(w)$ in this way, i.e., having determined the various unknown parameters in (14) or in (15), we substitute it into (11) and find $G(z, z^*)$. We can thus calculate the density of eigenvalues $\rho(r)$ explicitly for an arbitrary V.

We now turn to the disk-annulus phase transition. An important feature of this transition is that $F(w)$ is continuous through it. To see this we argue as follows: By tuning the couplings in V, we can induce a phase transition

from the disk phase into the annular phase, or vice versa. Note, of course, that we can parametrize any point in the disk phase either by the set of couplings in V or by the set of parameters $\{c_{-1}, c_0, \cdots c_{p-2}; b\}$ in (14). The "coordinate transformation" between these two sets of parameters is encoded in the asymptotic behavior of $F(w)$. Similarly, we can parametrize any point in the annular phase either by the set of couplings in V or by the set of parameters $\{c_0, c_1 \cdots c_{p-2}; a, b\}$ in (15). Due to the one-to-one relation (in a given phase, once we have established it is the stable one) between the couplings in V and the parameters in $F(w) - \frac{1}{2}V'(w)$ (namely, the c_n's and the locations of the branch points of $F(w)$), we can describe the disk-annulus transition in terms of the latter parameters (instead of the couplings in V). Clearly, the transition point is reached from the disk phase when $\rho(0) = 0$, that is, when c_{-1} in (14) vanishes:

$$c_{-1}^{crit} = 0\,. \tag{16}$$

Similarly, the transition point is reached from the annular phase when the lower branch point a in (15) vanishes. Thus, e.g., in a transition from the disk phase into the annular phase, $F_{disk}(w)$ in (14) would cross-over continuously into $F_{annulus}(w)$ in (15) through a critical form

$$F_{crit}(w) = \frac{1}{2}V'_{crit}(w) - \left(c_0^{crit} + c_1^{crit}\, w + \cdots + c_{p-2}^{crit}\, w^{p-2}\right)\sqrt{w(w - b^{crit})}\,. \tag{17}$$

The continuity of $F(w)$ through the transition was demonstrated explicitly in [11] for the quartic ensemble $V(\phi^\dagger\phi) = 2m^2\phi^\dagger\phi + g(\phi^\dagger\phi)^2$. In [1] we will present numerical results concerning this phase transition.

3 Boundaries and Boundary Values

Remarkably, with a minimal amount of effort, and based on the mere definition of $F(w)$ (Eq. (9)), we are able to derive simple expressions for the location of the boundaries of the eigenvalue distribution and also for the boundary values of $\rho(r)$ in terms of the moments

$$< \sigma^k >= \int_0^\infty \tilde{\rho}(\sigma)\sigma^k d\sigma \tag{18}$$

of $\tilde{\rho}(\sigma)$, the density of eigenvalues for a hermitean matrix problem. The technical details of this derivation will be given in [1].

The idea is simple. We rewrite our master formula (11) for $\gamma(r)$ as

$$wF(w) = \gamma \tag{19}$$

with

$$w = \frac{\gamma r^2}{\gamma - 1}\,. \tag{20}$$

Near the outer edge $r = R_{out}$ (either in the disk phase or in the annular phase) $\gamma \to 1-$, and thus $w \to -\infty$. Thus, to study the behavior near the outer edge, we simply expand $F(w)$ in (19) in powers of $1/w$. A generic term in this expansion is $< \sigma^k > /w^{k+1}$. For the class of models we are interested in here, all the moments $< \sigma^k >$, $k \geq 0$ are clearly finite.

In the annular phase, near the inner edge $r = R_{in}$, $\gamma \to 0+$, and thus $w \to 0-$. According to the discussion at the end of Section 2 (see Eq. (15) and the discussion above it), $a > 0$ in (12), and thus $F(w)$ is analytic in the domain $|w| < a$. Thus, to study the behavior near the inner edge, we expand $F(w)$ in (19) in powers of w. A generic term in this expansion is $w^k < 1/\sigma^{k+1} >$. From (15) we learn that in the annular phase $\tilde{\rho}_{annulus}(\sigma) \equiv \frac{1}{\pi}\mathrm{Im}F(\sigma - i\epsilon) = \mathrm{polynomial}(\sigma)\sqrt{(\sigma - a)(b - \sigma)}$, $0 < a < \sigma < b$ (and vanishes elsewhere.) Thus, all the negative moments $< 1/\sigma^k > \equiv \int_a^b (\tilde{\rho}(\sigma)/\sigma^k)d\sigma$, $k = 1, 2$ are finite.

From these expansions we readily find[1] that at the outer edge $r = R_{out}$,

$$R_{out}^2 =< \sigma >\,, \tag{21}$$

and

$$\rho(R_{out}) = \frac{2R_{out}^2}{< \sigma^2 > - < \sigma >^2}\,. \tag{22}$$

Namely, R_{out}^2 is simply the first moment of $\tilde{\rho}(\sigma)$, and the density $\rho(R_{out})$ is inversely proportional to the variance of σ! For the $\tilde{\rho}(\sigma)$ under consideration here, $< \sigma^2 >$, and consequently $\rho(R_{out})$, are always finite. Outside the boundary $\rho(r)$ vanishes identically, of course, and thus, $\rho(r)$ always "falls off a cliff" at the boundary, for all probability distributions of the form (1) with V polynomial. It would be thus interesting to study circularly invariant matrix ensembles $P(\phi^\dagger\phi)$ such that the eigenvalue distribution $\tilde{\rho}(\sigma)$ of $\phi^\dagger\phi$ has a finite $< \sigma >$ but an infinite $< \sigma^2 >$. Then $\rho(R_{out})$ would vanish.

Similarly, at the inner edge $r = R_{in}$ of the annular phase we find[1] from these expansions that

$$\frac{1}{R_{in}^2} = \langle\frac{1}{\sigma}\rangle\,, \tag{23}$$

and

$$\rho(R_{in}) = \frac{2R_{in}^{-6}}{<\sigma^{-2}> - <\sigma^{-1}>^2}\,. \tag{24}$$

Thus, R_{in}^{-2} is simply the σ^{-1} moment of $\tilde{\rho}(\sigma)$, and the density $\rho(R_{out})$ is inversely proportional to the variance of σ^{-1}. Since $< 1/\sigma >$ and $< 1/\sigma^2 >$ are finite, $\rho_{annulus}(\sigma)$ jumps from zero (in the inner void of the annulus) to a finite value at the inner edge R_{in}. Note, however, that when $a \to 0$, that is, in the annular to disk transition, $< 1/\sigma >$ remains finite, but $< 1/\sigma^2 >$ diverges like $1/\sqrt{a}$. Thus, from (23) we see that $R_{innner}(a = 0)$, the *critical* inner radius, is finite. The annulus strats up with a finite inner radius. Also, in this limit, we see from (24) that $\rho(R_{in})$ vanishes like $\sqrt{a}$. As we approach the annulus-disk transition, the discontinuity in $\rho(r)$ at the (finite) inner edge disappears.

We saw at the end of Section 2 (see Eqs. (14)-(17)) that $F(w)$ is continuous through the disk-annulus phase transition. Thus, our master formula $wF(w) = \gamma$ to determine $\gamma(r)$ (Eq. (19)) is also continuous through the transition. Consequently, $\rho(r) = (1/r)(d\gamma/dr)$ must remain continuous through the disk-annulus transition, and have (at the transition) the universal behavior described in the previous paragraph.

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