1. PHY9A Discussion 6 May 7, 2018

1.

i) Using the centripetal acceleration and Hooke's law,

$$m \cdot \frac{v_0^2}{\ell} = k(\ell - \ell_0)$$

$$\Leftrightarrow \quad 0 = \ell^2 - \ell_0 \cdot \ell - \frac{m v_0^2}{k}$$

$$\therefore \ell = \frac{\ell_0 + \sqrt{\ell_0^2 + \frac{4 m v_0^2}{k}}}{2} \qquad (\because \ell > 0).$$

$$\sqrt{\ell_0^2 + \frac{4 m v_0^2}{k}} = 2 v_0 \sqrt{\frac{m}{k}} \cdot \left(1 + \frac{k \ell_0^2}{4 m v_0^2}\right)^{1/2}$$

$$\simeq 2 v_0 \sqrt{\frac{m}{k}} \cdot \left(1 + \frac{1}{2} \cdot \frac{k \ell_0^2}{4 m v_0^2}\right) \qquad \left(\because \frac{k \ell_0^2}{4 m v_0^2} \simeq 3.5 \times 10^{-5} \ll 1.\right)$$

$$= 2 v_0 \sqrt{\frac{m}{k}} + \frac{\ell_0^2}{4 v_0} \sqrt{\frac{k}{m}}.$$

$$\therefore \ell \simeq \frac{\ell_0}{2} + v_0 \sqrt{\frac{m}{k}} + \frac{\ell_0^2}{8 v_0} \cdot \sqrt{\frac{k}{m}} \simeq 1272 \ [\mathrm{cm}].$$

ii)

$$E = KE + PE = \frac{1}{2} m v_0^2 + \frac{1}{2} k (\ell - \ell_0)^2$$

$$= \frac{1}{2} m v_0^2 + \frac{1}{2} m v_0^2 \left(1 - \frac{\ell_0}{\ell}\right)$$

$$= m v_0^2 \left(1 - \frac{\ell_0}{2\ell}\right)$$

$$\simeq 795.3 \ [\mathrm{J}].$$

2.

iii) $\tilde{l} := l|_{v_0 \to 2v_0} \approx \frac{l_0}{2} + 2v_0\sqrt{\frac{m}{k}} + \frac{l_0^2}{16v_0}\sqrt{\frac{k}{m}} \approx 2537$ [cm].

$\therefore \tilde{l} \approx 2l$.

$$\tilde{E} = m(2v_0)^2\left(1 - \frac{l_0}{2\tilde{l}}\right) = 4mv_0^2\left(1 - \frac{l_0}{2\tilde{l}}\right)$$

$$\approx 3181 \text{ [J]}.$$

$\therefore \tilde{E} \approx 4E$.

2.

(a) Integrating along the y-axis,

$$U \equiv -\int \mathbb{F} \cdot d\mathbb{s} = C \int dy = Cy + U_0.$$

(eg: $C = mg \Rightarrow F = -mg\,\mathbb{e}_y$, $U = mgy + U_0$.)

(b) Integrating into the radial direction,

$$U \equiv -\int \mathbb{F} \cdot d\mathbb{s} = -C \int \frac{1}{r^2} dr = \frac{C}{r} + U_0.$$

(c) Using $\nabla f(r) = \mathbb{e}_r \cdot \frac{df}{dr}$,

$$\mathbb{F} \equiv -\nabla U = -C\,\mathbb{e}_r \cdot \frac{d}{dr}\left(\frac{1}{r^3}\right) = \frac{3C}{r^4}\,\mathbb{e}_r.$$

(cf: Quadrupole potential.)

3.

(a) $$\mathbf{F} = -\nabla U = -C\,\mathbf{e}_r \cdot \frac{d}{dr}\left(\frac{e^{-r/r_0}}{r}\right)$$

$$= -C\,\mathbf{e}_r\left(-\frac{1}{r_0}\cdot\frac{e^{-r/r_0}}{r} - \frac{e^{-r/r_0}}{r^2}\right)$$

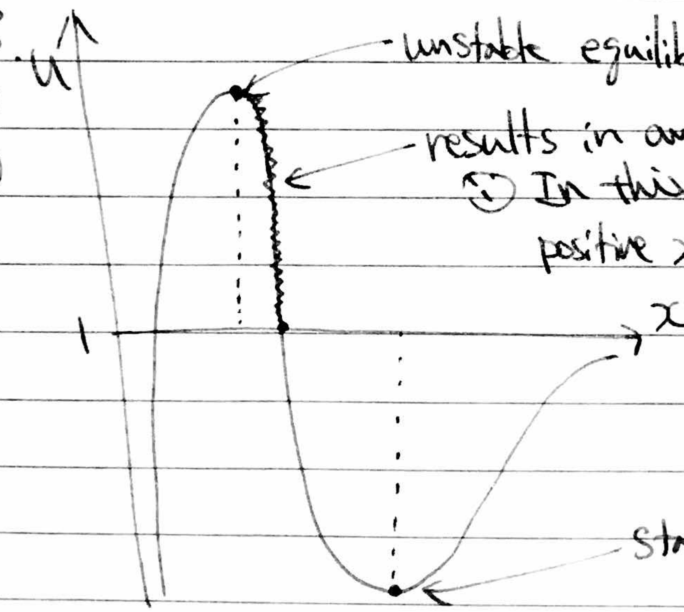
$$= C\,\frac{e^{-r/r_0}}{r}\left(\frac{1}{r}+\frac{1}{r_0}\right)\cdot \mathbf{e}_r\,.$$

(cf: Yukawa potential.)

3

i)

ii)



unstable equilibrium

results in an unbounded motion.

$\because$ In this region, the object starts to move into positive x direction with $E = PE \geq 1$.

At later time, $PE \leq 1$, and thus $KE \geq 0$ since $1 \leq KE + PE$.

(with $KE = 0$ only at $x \to \infty$ when the initial $PE = 1$).

stable equilibrium.

ex) This is a potential energy of a massive object orbiting around a Schwarzschild black hole with high enough angular momentum.