1//

Mass on spring with damping:

$$m\frac{d^2x}{dt^2} = -kx - bv$$

$b=0$: $\quad \frac{d^2x}{dt^2} = -\frac{k}{m}x$

$x = A\sin\omega t \;;\; B\cos\omega t \qquad \omega = \sqrt{\frac{k}{m}}$

$b \neq 0$: Useful to work with complex exponentials

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \ldots$$

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

Seems like e^x has something to do with sin and cos but weird "$-$" get in the way. Try this

$$e^{ix} = 1 + ix + \frac{1}{2}(ix)^2 + \frac{1}{6}(ix)^3 + \frac{1}{24}(ix)^4 + \frac{1}{120}(ix)^5 + \ldots$$

$$i^2 = -1$$

$$e^{ix} = 1 + ix - \frac{1}{2}x^2 - \frac{1}{6}ix^3 + \frac{1}{24}x^4 + \frac{1}{120}ix^5$$

$$= \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \ldots\right) + i\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \ldots\right)$$

$$\boxed{e^{ix} = \cos x + i\sin x \;!!}$$

2/1

Useful for solving damped oscillator

Guess the sol'n is $x(t) = Ae^{i\omega t}$

$$m\frac{d^2x}{dt^2} = -kx - b\underbrace{\frac{dx}{dt}}_{v}$$

$$-m(i\omega)^2\,\cancel{Ae^{i\omega t}} = -k\,\cancel{Ae^{i\omega t}} - b(i\omega)\,\cancel{Ae^{i\omega t}}$$

$$-m\omega^2 = -k - i\omega b$$

$$m\omega^2 - i\omega b - k = 0 \quad \longleftarrow \text{quadratic eqn}$$

$$\omega = \frac{ib \pm \sqrt{(-ib)^2 + 4km}}{2m} \quad \longleftarrow \text{quadratic formula}$$

still valid even if coefficients complex!

$$\omega = \frac{ib}{2m} \pm \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

Plug back into $Ae^{i\omega t}$

$$x(t) = A\underbrace{e^{-bt/2m}}_{\text{decaying exponential}}\;\underbrace{e^{\pm i\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}\,t}}$$

sin and cos except ω is reduced from $\sqrt{\frac{k}{m}}$ to $\sqrt{\frac{k}{m} - b^2/4m^2}$

3/1

General Solution:

$$\boxed{x(t) = e^{-bt/2m}\left\{A\cos\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}\,t + B\sin\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}\,t\right\}}$$

A, B are determined by $x(t=0)$ and $v(t=0)$

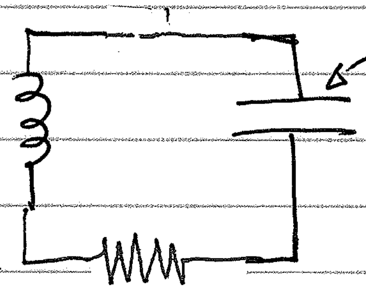
Electric circuit analogies:

$$m\frac{d^2x}{dt^2} + kx + b\frac{dx}{dt} = 0$$

$$L\frac{d^2Q}{dt^2} + \frac{1}{C}Q + R\frac{dQ}{dt} = 0$$

Kirchoff's Law: Sum of voltages around closed loop is zero

charge Q on plates



Inductor L

Voltage $L\frac{dI}{dt} = L\frac{d^2Q}{dt^2}$

capacitor C

Voltage Q/C

resistor R

voltage $IR = R\frac{dQ}{dt}$

"Dictionary"

mass-spring	circuit
$x(t)$	$Q(t)$
$v(t)$	$I(t)$
m	L
k	$1/C$
b	R

$$\boxed{Q(t) = e^{-Rt/2L}\left\{A\cos\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\,t + B\sin\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\,t\right\}}$$

A//

We want significant shift in ω so we can observe it easily, eg

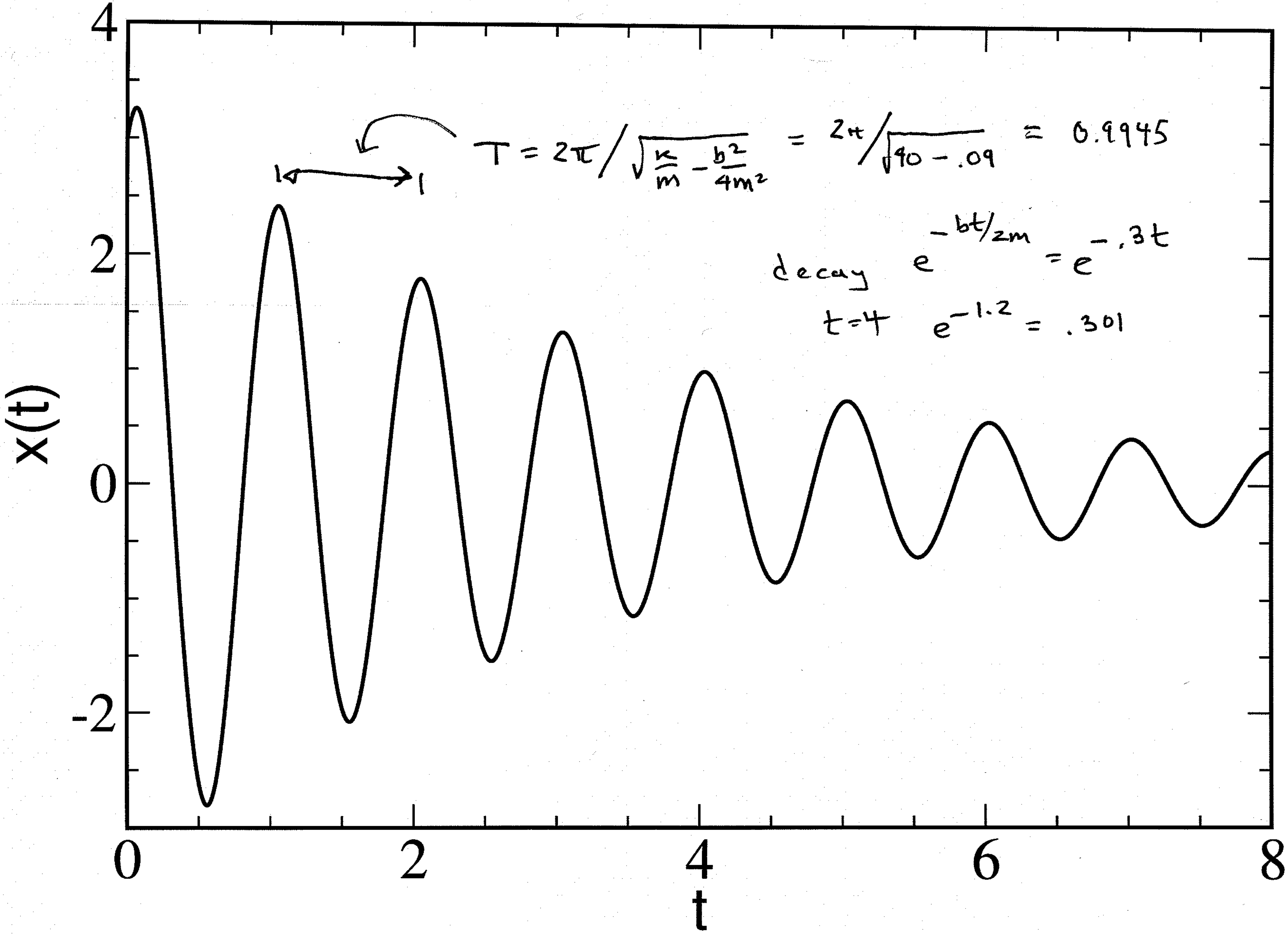
$$b^2/4m^2 = 1/4 \; k/m$$

But also want oscillations eg $\overset{\tau}{2m/b} = \ell \overset{T}{2\pi/\omega}$.

$\ell \sim 6$ would be nice

$$\left.\begin{array}{ll} b^2 = km & b = \sqrt{km} \\ b = m\omega/\ell\pi = \dfrac{\sqrt{km}}{\ell\pi} & \end{array}\right\} \text{inconsistent!}$$

x=3 v0=8 m=0.25 k=10 b=0.15 dt=0.002 N=4000



1/1

Projectile Motion / Range formula

Standard (no air resistance) analysis:

$X_0 = 0 \qquad y_0 = 0$ Assumed

$X = V_0 \cos\theta \, t$

$y = V_0 \sin\theta \, t - gt^2/2$

Hits ground at $y = 0 = V_0 \sin\theta \, t - \frac{1}{2} g t^2$

$t = 2 V_0 \sin\theta / g$

at which point $X = V_0 \cos\theta \frac{2 V_0 \sin\theta}{g}$

$\Rightarrow R = \frac{V_0^2}{g} 2 \sin\theta \cos\theta = \frac{V_0^2}{g} \sin 2\theta$

Since sine is maximum at $90°$ want $2\theta = 90°$

so R is biggest at $\theta = 45°$.

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Try now with air resistance

(1) $$\frac{dv_x}{dt} = -by\,v_x$$

(2) $$\frac{dv_y}{dt} = -g - by\,v_y$$ ← nonlinear; dead duck!

To solve (1) we need $y(t)$, so work on (2) first