PDE1

Partial Differential Equations

* Algebraic Eqn

An equation obeyed by a number x, eg

$$2x-6=0 \Rightarrow x=3$$

$$ax^2+bx+c=0 \Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Goal: find number(s) obeying the eqn

~ Diophantine: restricted to integers, eg pythagorean triples.

* Differential Eqn:

An equation obeyed by a function $x(t)$ of a single variable

$$m\frac{d^2}{dt^2}x(t)=-kx$$

Goal: find function(s) obeying the differential eqn

$$x(t)=A\sin\sqrt{\frac{k}{m}}\,t+B\cos\sqrt{\frac{k}{m}}\,t$$

PDE 2

"Degree" of algebraic eqn $\Rightarrow$ # of solns

$ax^1 + b = 0$: degree 1 $\rightarrow$ 1 solution

$ax^2 + bx + c = 0$: degree 2 $\rightarrow$ 2 solutions

etc

Same for differential equation : count highest # of derivatives

$$\frac{dx}{dt} = 4x \qquad x(t) = Ae^{4t}$$

first derivative $\rightarrow$ degree 1

one solution

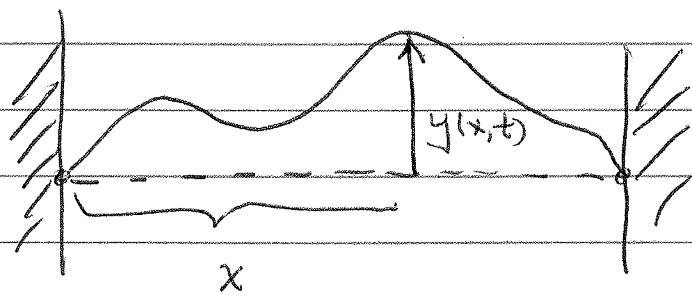
$$m\frac{d^2x}{dt^2} = -kx$$

degree 2

two solutions

PDE 3

Vibrating string $y(x,t)$



Wave eqn $\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0$

Laplace eqn (later) Schroedinger Eqn (later)

Diffusion eqn

$\rho(x,t)$ = density of smoke

$$\frac{\partial \rho(x,t)}{\partial t} = D\, \partial^2 \rho / \partial x^2$$

$\rho(x,t)$ = temperature

diffusion constant

large D → rapid spreading

PDE4

Numerical first derivative

$$\frac{df}{dx} = \frac{f(x+dx) - f(x)}{dx}$$

Second derivative

$$\frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{df/dx\big|_x - df/dx\big|_{x-dx}}{dx}$$

$$= \frac{1}{dx}\left\{ \frac{f(x+dx) - f(x)}{dx} - \frac{f(x) - f(x-dx)}{dx} \right\}$$

$$= \frac{f(x+dx) - 2f(x) + f(x+dx)}{dx^2}$$

Check

$$f(x) = mx + b$$

$$f(x+dx) = mx + m\,dx + b$$

$$f(x-dx) = mx - m\,dx + b$$

$f(x+dx) - 2f(x) + f(x-dx) = 0$! ✓

Check

$$f(x) = ax^2$$

$$f(x+dx) = a(x^2 + 2x\,dx + dx^2)$$

$$f(x-dx) = a(x^2 - 2x\,dx + dx^2)$$

$$f(x+dx) - 2f(x) + f(x-dx) = 2a\,dx^2 \rightarrow \frac{d^2f}{dx^2} = 2a$$ ✓✓

PDE5

Alternate derivation (Taylor Thm)

$$f(x+dx) = f(x) + \frac{df}{dx}dx + \frac{1}{2}\frac{d^2f}{dx^2}(dx)^2 + \dots$$

$$f(x-dx) = f(x) - \frac{df}{dx}dx + \frac{1}{2}\frac{d^2f}{dx^2}(dx)^2 + \dots$$

$$f(x+dx) + f(x-dx) = 2f(x) + \frac{d^2f}{dx^2}dx^2$$

SAME!

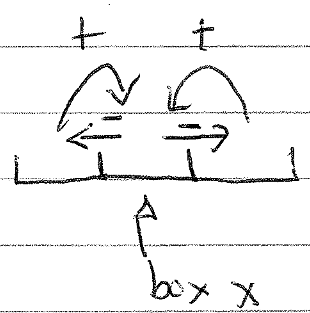
Back to Diffusion Eqn

partial derivative: change one variable, keep other fixed!

$$\frac{\rho(x,t+dt)-\rho(x,t)}{dt} = D\,\frac{\rho(x+dx,t)-2\rho(x,t)+\rho(x-dx,t)}{dx^2}$$

$\frac{\partial}{\partial t}$: x fixed, t varies $\qquad \frac{\partial^2}{\partial x^2}$: t fixed, x varies

$$\rho(x,t+dt) = \rho(x,t) + \left(\frac{D\,dt}{dx^2}\right)\left(\rho(x+dx,t)-2\rho(x,t)+\rho(x-dx,t)\right)$$



Picture explains factor of 2 and also signs!