PROBLEM SET 3 Due Friday November 8

Physics 240A– FALL 2019

[**1.**] Consider a one-dimensional chain of identical atoms. The springs between them alternate in strength between values K_1 and K_2.
a) Find the vibrational frequencies as a function of wave number q. Study the low q limit and find the sound velocity.
b) Discuss the physical meaning of the two branches. Sketch the way the atoms move in both cases.
c) Discuss the dispersion and the normal modes for $K_1 \gg K_2$.
d) Discuss the limit $K_1 \approx K_2$ and compare with the homogeneous chain where all springs are identical (see class).

[**2.**] Consider a one-dimensional chain of identical atoms of mass M. The springs are not only between nearest neighbors but between all pairs of atoms. Thus, the elastic energy reads

$$E_{\rm el} = \frac{1}{2}\sum_n \sum_{m>0} K_m \left(x_n - x_{n+m}\right)^2$$

where x_n is the displacement of atom n.
a) Find the dispersion relation, i.e. the vibrational frequency ω as a function of wave number q.
b) Assume $K_m = K_0/m^p$ with $p > 1$ a parameter controlling how rapidly the interaction drops of with distance. Study the long-wavelength limit of the dispersion relation for $p > 3$. Determine the sound velocity.
c) Investigate the long-wavelength limit of the dispersion relation for $1 < p < 3$. Show that one gets "anomalous sound", i.e., the frequency is not proportional to the wavenumber. (Hint: You may want to approximate the m-sum by an integral.)

[**3.**] The specific heat C of a two level system goes to zero at high T, but the specific heat of a classical (or quantum) oscillator remains finite no matter how high T gets. What property of the energy levels of a system determines whether C vanishes at high T or not? Interpret your answer in terms of the formula $dE = C\,dT$.

[**4.**] The specific heat C of a two level system goes to zero exponentially at low T. How does the specific heat C of a quantum oscillator of frequency ω_0 go to zero at low T? In class we saw that a 3D set of oscillators with a linear dispersion relation $\omega(q) = vq$ has a low T specific heat which vanishes as a power law $C(T) = AT^3$. What property of the energy levels of a system determines whether $C(T)$ is exponentially small at low T vs some less rapidly decaying function like T^α?

Note: The last two problems are *numeric*. For the first, you will need to be able to do integrals numerically. For the second you will need to be able to diagonalize a matrix by calling some appropriate mathematical library. I will post a way to do both of these using C programs on the course website, but feel free to use whatever software/language you like. Talk to me or Ben if you need help.

[**5.**] We showed in class that the specific heat of a 1D classical harmonic oscillator with energy $E(x,p) = kx^2/2 + p^2/2m$ is $C(T) = k_{\rm B}$. Compute and plot the specific heat of an *anharmonic* oscillator $E(x,p) = kx^2/2 + \gamma x^4/4 + p^2/2m$. Choose $k = 0.7$ and $\gamma = 0.0, 0.1, 0.2, 0.5$. Comment on your results. Does γ increase or decrease C? What happens to C at low and high T? Is there any way to have predicted some of these answers?

One approach is to solving this problem is to use the fluctuation form for the specific heat

$$\begin{aligned} C &= \beta^2 \left(\langle E^2 \rangle - \langle E \rangle^2 \right) \\ \langle E \rangle &= \frac{1}{Z} \int dx dp \, E(x,p) \, \exp[-E(x,p)/k_{\rm B}T] \\ \langle E^2 \rangle &= \frac{1}{Z} \int dx dp \, E(x,p)^2 \, \exp[-E(x,p)/k_{\rm B}T] \\ Z &= \int dx dp \, \exp[-E(x,p)/k_{\rm B}T] \end{aligned}$$

Then do the integrals numerically. Can the integrals be done analytically?

[**6.**] Consider a linear (1D) mass-spring system in which one of the springs (a "defect") has a value k_* which is different from all the others, which have value k. Assume all masses m are equal. As in the isotropic case, the normal mode frequencies ω^2 are determined by writing down Newtson's equations $F = ma$ for all the masses. If you assume, as usual, $x_n(t) = a_n e^{i\omega t}$, write down the matrix which, when diagonalized, has eigenvalues which are the squares ω^2 of the normal mode frequencies. Now diagonalize the matrix (numerically). Choose number of masses $N = 64$, spring constant $k = 1.9$, mass $m = 1.1$, and defect spring $k_* = 5.2$.

What do you notice about the eigenvalues?

Compute the participation ratios (see below) of the 64 eigenvectors. What do you notice? Interpret your result physically.

This is actually a really important problem which arises in many contexts. If you have time, play around with your code a bit further. What happens as $k_* \to k$? What happens if $k_* < k$? Think about why $k_* < k$ might be different from $k_* > k$.

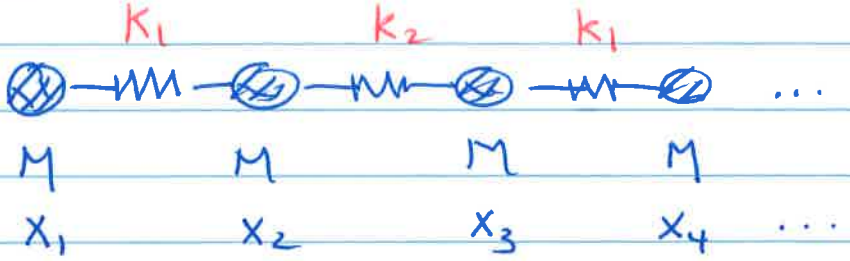
Definition: Given a normalized vector $\vec{v}$ with components v_n, $n = 1, 2, \cdots N$, the participation ratio

$$\mathcal{P} = \left(\sum_n v_n^4 \right)^{-1}$$

provides an estimate of the number of components of $\vec{v}$ which are of significant size. (See class discussion.)

P240A PS 3

(a)



ℓ odd: $M\ddot{x}_\ell = -k_1(x_\ell - x_{\ell+1}) - k_2(x_\ell - x_{\ell-1})$

ℓ even: $M\ddot{x}_\ell = -k_2(x_\ell - x_{\ell+1}) - k_1(x_\ell - x_{\ell-1})$

Guess soln $x_\ell(t) = a_\ell e^{i\omega t}$ ℓ odd

$x_\ell(t) = b_\ell e^{i\omega t}$ ℓ even

$-M\omega^2 a_\ell = -k_1(a_\ell - b_{\ell+1}) - k_2(a_\ell - b_{\ell-1})$

$-M\omega^2 b_\ell = -k_2(b_\ell - a_{\ell+1}) - k_1(b_\ell - a_{\ell-1})$

Then also guess $a_\ell = a_0 e^{iq\ell}$ $b_\ell = b_0 e^{iq\ell}$

$-M\omega^2 a_0 = -k_1(a_0 - b_0 e^{iq}) - k_2(a_0 - b_0 e^{-iq})$

$-M\omega^2 b_0 = -k_2(b_0 - a_0 e^{iq}) - k_1(b_0 - a_0 e^{-iq})$

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$$\begin{pmatrix} k_1+k_2 & -k_1 e^{iq} - k_2 e^{-iq} \\ -k_2 e^{iq} - k_1 e^{-iq} & k_1+k_2 \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = M\omega^2 \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$$

↑ notice this matrix Hermitian $\Rightarrow$ real eigenvalues!

$$(k_1+k_2-\lambda)^2 - \left(k_1^2+k_2^2+k_1k_2(e^{2iq}+e^{-2iq})\right) = 0$$

$$\lambda^2 - 2\lambda(k_1+k_2) + [\,k_1^2+2k_1k_2+k_2^2 - k_1^2-k_2^2 - 2k_1k_2\cos 2q\,] = 0$$

$$\lambda^2 - 2\lambda(k_1+k_2) + 2k_1k_2(1-\cos 2q) = 0$$

$$\lambda = M\omega^2 = \frac{2(k_1+k_2) \pm 2\sqrt{(k_1+k_2)^2 - 2k_1k_2(1-\cos 2q)}}{2}$$

$$\boxed{\omega^2 = \frac{k_1+k_2 \pm \sqrt{k_1^2+k_2^2+2k_1k_2\cos 2q}}{M}}$$

what are allowed q values? Looking at the end of chain eqns

$$l=1 \qquad M\ddot{x}_1^{oo} = -k_1(x_1-x_2) - k_2(x_1-x_0)$$

{ identify with x_N (and assume N even)

PBC restrict $q = \frac{2\pi}{N}\left\{-\frac{N}{2}+1, \ldots \frac{N}{2}\right\}$

but because $2q$ appears in eqn for ω^2

the eigenvalues are duplicated over this range so

instead $q = \frac{2\pi}{N}\left\{-\frac{N}{4}+1, \ldots \frac{N}{4}\right\}$

$N/2$ values for q each with 2 ω^2

$\Rightarrow$ N eigenvalues ✓

If q is small $\cos 2q \approx 1 - \frac{1}{2}(2q)^2 \approx 1 - 2q^2$

$$\sqrt{k_1^2 + k_2^2 + 2k_1k_2\cos 2q} \approx \sqrt{(k_1+k_2)^2 - 4k_1k_2q^2}$$

$$= (k_1+k)\left[1 - \frac{4k_1k_2}{(k_1+k_2)^2}q^2\right]^{1/2}$$

$$\approx (k_1+k_2)\left[1 - \frac{2k_1k_2q^2}{(k_1+k_2)^2}\right]$$

$$\omega^2 = \frac{1}{M}\left\{(k_1+k_2) \pm (k_1+k_2)\left(1 - \frac{2k_1k_2q^2}{(k_1+k_2)^2}\right)\right\}$$

$$\omega_-^2 \approx \frac{1}{M}\frac{2k_1k_2}{(k_1+k_2)}q^2$$

$$\omega_+^2 \approx \frac{2(k_1+k_2)}{M}$$

The ω_-^2 is acoustic $\omega_- = vq$

with $v = \sqrt{\frac{2k_1k_2}{M(k_1+k_2)}}$

This has dimensions of $1/T$ (frequency) rather than velocity. This is because when we write e^{iql} with l an integer our q is dimensionless.

To "restore units" we could write e^{iqla} where "a" is the lattice constant. Then q will have units $1/L$.

This just amounts to replacing q by qa throughout the calculation

$$v = \sqrt{\frac{2k_1k_2}{M(k_1+k_2)}}\, a$$

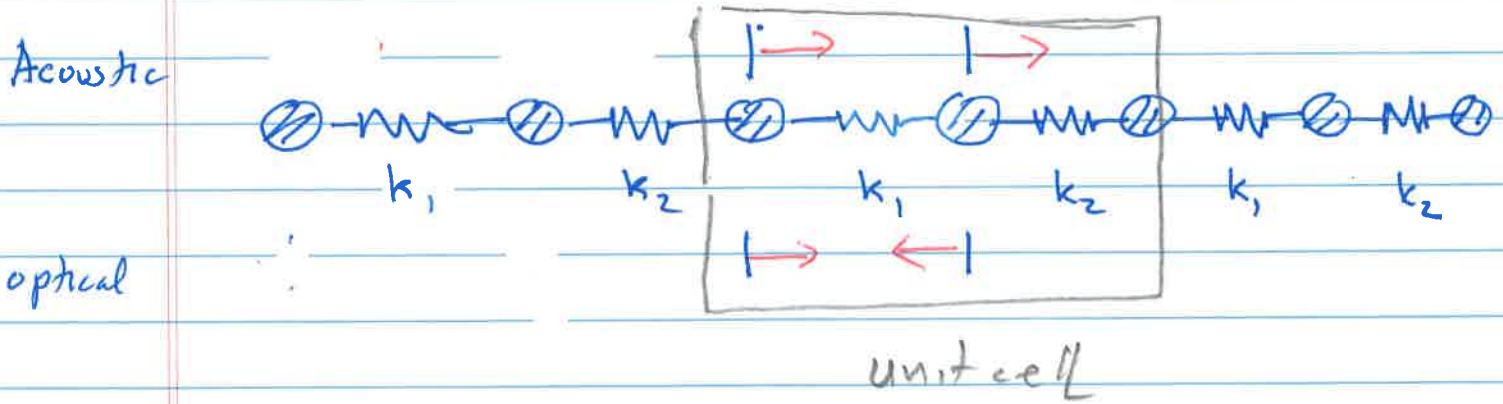
a: distance between masses.

NB. In some solutions to this problem "a" is defined as the distance between two masses, ie the size of a unit cell.

(b) We already identified ω_- as acoustic branch

ω_+ is of course optic branch.

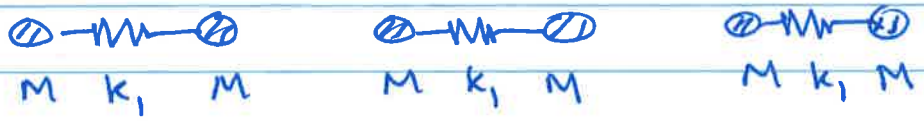
Sketching how masses move is a matter of examining the eigenvectors associated with these eigenvalues. You can do that math or else recall the general principle that higher energy $\leftrightarrow$ more nodes. So we expect optic branch to have masses in unit cell to move opposite to each other and in acoustic branch to move together in same direction.



(c) $k_1 >> k_2$ $\qquad \omega_+^2 = \frac{2(k_1+k_2)}{M} \approx \frac{2k_1}{M}$

$$\omega_-^2 = \frac{2(k_1 k_2)}{(k_1+k_2)M} q \approx \frac{2k_2}{M} q$$

The ω_+ (optic) modes correspond to a bunch of independent 2 mass molecules:



The ω_- (acoustic) modes correspond to the two masses joined by strong k_2 moving as a unit of mass $2M$

(d) $k_1 = k_2 = k \qquad k_1^2 + k_2^2 + 2k_1 k_2 \cos 2q$

$\rightarrow 2k^2(1+\cos 2q)$

$\rightarrow 4k^2 \cos^2 q$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$1 = \cos^2\theta + \sin^2\theta$

$1 + \cos 2\theta = 2\cos^2\theta$

$$\omega^2 = \frac{2k \pm 2k\cos q}{M} = \frac{2k}{M}(1 \pm \cos q) \quad \text{W}$$

$$E = \frac{1}{2} \sum_n \sum_{m>0} k_m (x_n - x_{n+m})^2$$

$$= \frac{1}{2} \{ k_1 (x_1 - x_2)^2 + k_2 (x_1 - x_3)^2 + k_3 (x_1 - x_4)^2 + \cdots$$

$$+ k_1 (x_2 - x_3)^2 + k_2 (x_2 - x_4)^2 + k_3 (x_2 - x_5)^2 + \cdots$$

$$+ k_1 (x_3 - x_4)^2 + k_2 (x_3 - x_5)^2 + k_3 (x_3 - x_6)^2 + \cdots$$

Looking at x_3 we see it is connected to x_1 by $k_2 (x_1 - x_3)^2$ and also to x_5 by $k_2 (x_3 - x_5)^2$

In general x_ℓ will be connected to $x_{\ell \pm n}$ by k_n

and the Equation of Motion is

$$M \ddot{x}_\ell = -k_1 (x_\ell - x_{\ell+1}) - k_1 (x_\ell - x_{\ell-1})$$

$$- k_2 (x_\ell - x_{\ell+2}) - k_2 (x_\ell - x_{\ell-2}) \cdots$$

Making the usual assumption $x_\ell(t) = e^{iq\ell} e^{i\omega t}$

$$M\omega^2 = 2k_1 (1 - \cos q) + 2k_2 (1 - \cos 2q)$$

$$+ 2k_3 (1 - \cos 3q) + \cdots$$

and, ∞ on page 1-6 $\quad 1 - \cos 2\theta = 2 \sin^2 \theta$

$$M\omega^2 = \sum_{n>0} 4k_n \sin^2(nq/2)$$

(b) Long wavelength → q small → $\sin^2 \frac{nq}{2} \approx n^2q^2/2$

$$M\omega^2 = \sum_{n>0} 4\frac{k_0}{n^p}\frac{n^2q^2}{2} = \sum_{n>0} \frac{2k_0}{n^{p-2}} q^2$$

$$v_{sound} = \sqrt{\frac{1}{m}\sum_{n>0} 2k_0/n^{p-2}}\; a$$

a ← lattice spacing (see page 1-4)

sum converges if $p>3$

(c) If $p<3$ we don't want to write $\sin^2 \frac{nq}{2} \approx n^2q^2/2$

$$M\omega^2 = \sum_{n>0} 4k_n \sin^2 \frac{nq}{2}$$

If q is small $nq/2$ closely spaced and approximate sum as integral

$$M\omega^2 \approx \int_0^\infty dn\; 4\frac{k_0}{n^p}\sin^2\frac{nq}{2} \qquad x = \frac{nq}{2} \quad n = \frac{2x}{q}$$

$$= \int_0^\infty \frac{2dx}{q}\, 4k_0\left(\frac{q}{2}\right)^p \frac{1}{x^p}\sin^2 x$$

ω^2 proportional to q^{p-1} →

$$= \frac{8k_0\, q^{p-1}}{2^p}\int_0^\infty dx\, \frac{\sin^2 x}{x^p}$$

nice and convergent as long as $p>1$

If a system has a maximal energy level (or, more precisely, a maximal energy difference) then C(T) will go to zero at large T.

One way to view this is in terms of $dE = C\,dT$. If there is a maximum energy, at some point as you change dT the energy can no longer respond: $dE \to 0$ even though $dT \neq 0$. Hence $C = 0$.

Another way to look at this is via the identity

$$C \equiv \frac{d\langle E\rangle}{dT} = \frac{\langle E^2\rangle - \langle E\rangle^2}{T^2}$$

If the energy levels have a maximal value the numerator has an upper bound, so as $T \to \infty$ the denominator drives C to zero. (If there is no maximum level the numerator can increase indefinitely as T increases, and "keep pace" with the denominator.)

The quantum oscillator has an energy gap $\hbar\omega$ between the ground state $E_0 = \frac{1}{2}\hbar\omega$ and the first excited state $E_1 = \frac{3}{2}\hbar\omega$. In such a situation, C will behave exponentially: $C \sim e^{-\Delta/T}$

If you want to examine this more mathematically

$$\langle E \rangle = \frac{E_1 e^{-\beta E_1} + E_2 e^{-\beta E_2} + \dots}{e^{-\beta E_1} + e^{-\beta E_2} + \dots} = \frac{E_1 + E_2 e^{-\beta(E_2 - E_1)} + \dots}{1 + e^{-\beta(E_2 - E_1)} + \dots}$$

Defining $\Delta = E_2 - E_1$ we get

$$\langle E \rangle = E_1 + E_2 e^{-\beta\Delta} + \dots$$

$$C = d\langle E \rangle / dT = d\beta/dT \; d\langle E \rangle / d\beta = -\frac{1}{T^2}(-\Delta) E_2 e^{-\beta\Delta}$$

$$C \sim \beta^2 E_2 \Delta e^{-\beta\Delta}$$

Note that as $\beta \to \infty$ $e^{-\beta\Delta}$ easily overwhelms β^2

If $C(T)$ is <u>not</u> to decay exponentially there must be no energy gap: there must be excited state energies arbitrarily close to E_0. This clearly true, for example, for a classical oscillator where

$$E(x,p) = \tfrac{1}{2}k^2 + p^2/2m$$

can be as close as you like to $E_0 = 0$ just by making x, p small.

In general, the crucial quantity is the density of states $N(E)$. We will discuss this soon, but $N(E)$ is the number of states with energy E so that

$$Z = \int N(E)\, e^{-\beta E}\, dE$$

$$\langle E \rangle = Z^{-1} \int N(E)\, E\, e^{-\beta E}\, dE$$

suppose there is an $E_{min} = E_0$ and $N(E) = (E - E_0)^p$

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$$Z = \int_{E_0}^{\infty} (E - E_0)^p e^{-\beta E} \, dE \qquad x \equiv \beta(E - E_0)$$

$$= \int_0^{\infty} \left(\frac{x}{\beta}\right)^p e^{-x} e^{\beta E_0} \, dE$$

$$Z = \frac{1}{\beta^p} e^{+\beta E_0} \underbrace{\int_0^{\infty} x^p e^{-x} \, dx}_{p!} = \frac{p!}{\beta^p} e^{+\beta E_0}$$

$$\langle E \rangle = Z^{-1} \int_{E_0}^{\infty} E \, (E - E_0)^p e^{-\beta E} \, dE$$

$$= Z^{-1} \int_0^{\infty} \left(\frac{x}{\beta} + E_0\right)\left(\frac{x}{\beta}\right)^p e^{-x} e^{\beta E_0} \, dE$$

$$= E_0 + Z^{-1} \frac{1}{\beta^{p+1}} e^{\beta E_0} \underbrace{\int_0^{\infty} x^{p+1} e^{-x} \, dx}_{(p+1)!}$$

$$= E_0 + \frac{\beta^p e^{-\beta E_0}}{p!} \frac{1}{\beta^{p+1}} e^{\beta E_0} (p+1)!$$

$$= E_0 + \frac{(p+1)!}{p!} T$$

$$c = \frac{(p+1)!}{p!}$$

Note that because there are no "cross terms" between x and p, we can treat them independently.

That is, $E(x,p) = E_1(x) + E_2(p)$

$$Z = \int dx \int dp\, e^{-\beta E} = \int dx\, e^{-E_1(x)\beta} \cdot \int dp\, e^{-E_2(p)\beta}$$

$$Z = Z_x Z_p$$

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \ln Z = -\frac{\partial}{\partial \beta} (\ln Z_x + \ln Z_p)$$

$$\langle E \rangle = \langle E \rangle_x + \langle E \rangle_p$$

{ just compute them independently!

We know $\langle E \rangle_p = \frac{1}{2} T$ ($k_B = 1$ in all I do here)

$$C_p = \frac{1}{2}$$

So in the next two pages I just compute $\langle E \rangle_x$ and C_x.

5-2

```
#include <stdio.h>
#include <math.h>

int main(void)
{
    FILE * fileout;
    fileout=fopen("C.dat","w");

    double k,x,dx,g;
    double E,E2,Z,energy,W,T,T0,dT,beta,C;
    int n,i,nT,iT;
    printf("\nEnter n:  ");
    scanf("%i",&n);
    printf("Enter dx: ");
    scanf("%lf",&dx);
    printf("Enter k,gamma,nT,T0,dT : ");
    scanf("%lf %lf %i %lf %lf",&k,&g,&nT,&T0,&dT);
    beta=1.0/T;

    for (iT=0;iT<nT+1;iT=iT+1)
    {

         T=T0+dT*iT;
         beta=1.0/T;

         E =0.0;
         E2=0.0;
         Z =0.0;

         for (i=-n;i<n+1;i=i+1)
         {
             x=dx*i;
             energy=0.5*k*x*x + 0.25*g*x*x*x*x;
             W=exp(-beta*energy);
             Z = Z+                W;
             E = E+       energy*W;
             E2=E2+energy*energy*W;
         }
         E =E /Z;
         E2=E2/Z;
         C =beta*beta*(E2-E*E);
//       printf(" <E>=  %12.4lf\n",E);
//       printf("  C =  %12.4lf\n",C);
         fprintf(fileout,"%12.4lf %12.4lf %12.4lf \n",T,E,C);

     }

     fclose(fileout);
     return 0;
}
```

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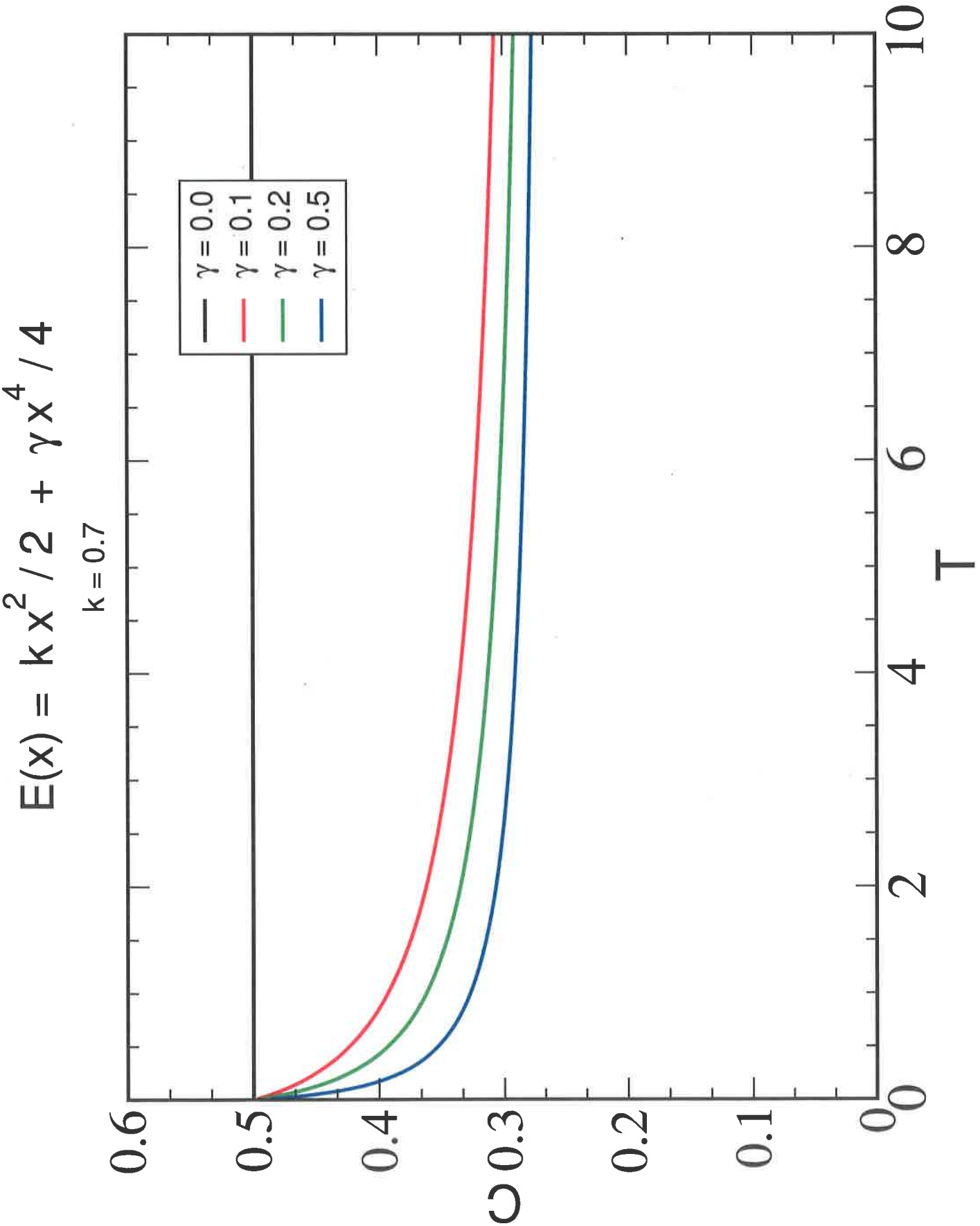

N=64 K=1.9 Kp=5.2

mode	norm	P
1	0.999322	2.213135
2	1.000000	64.000000
3	0.999516	41.668301
4	0.999502	42.279011
5	0.999258	41.744463
6	0.999662	41.154376
7	1.000916	42.626678
8	1.000678	42.629741
9	0.999306	41.956241
10	0.999720	42.641472
11	0.999720	42.641472
12	0.997904	42.371650
13	1.000916	42.626678
14	0.999720	42.641472
15	1.000142	41.966838
16	0.999860	42.242717
17	1.000916	42.626678
18	1.000916	42.626678
19	1.000100	42.307145
20	1.000090	42.264398
21	0.999720	42.641472
22	1.000916	42.626678
23	0.999980	42.065590
24	0.998798	42.325851
25	0.999720	42.641472
26	1.000028	41.737553
27	0.996768	42.917952
28	1.000476	42.139626
29	1.000916	42.626678
30	1.000474	41.903098
31	1.000464	42.823311
32	1.000916	42.626678
33	1.000916	42.626678
34	0.996768	42.917952
35	0.999848	42.253327
36	0.996768	42.917952
37	1.000916	42.626678
38	0.998682	42.027396
39	1.000644	42.168984
40	0.999666	41.688054
41	0.998208	41.801594
42	0.999322	42.112825
43	1.000916	42.626678
44	0.996768	42.917952
45	0.999964	41.827649
46	0.999720	42.641472
47	0.999720	42.641472
48	0.998780	42.203804
49	1.000916	42.626678
50	1.000636	42.117452
51	1.000916	42.626678
52	0.999372	41.804413
53	0.999720	42.641472
54	1.000916	42.626678
55	1.000424	42.087316
56	1.000117	41.834229
57	1.000916	42.626678
58	1.000916	42.626678
59	1.000152	42.138274
60	0.998176	42.967544
61	0.998176	42.967544
62	0.999878	42.494024

when defect spring $k' > K$ there is a localized mode only ≈ 2 masses (the ones connected by the defect spring k') participate

all other modes have most of the $N = 64$ masses vibrating

eigenvalues: N=64 K=1.9 Kp=5.2

0.000
0.018
0.019
0.073
0.074
0.164
0.167
0.289
0.295
0.449
0.458
0.640
0.653
0.863
0.880
1.113
1.135
1.389
1.416
1.689
1.722
2.009
2.047
2.346
2.390
2.697
2.748
3.059
3.115
3.428
3.490
3.800
3.868
4.172
4.246
4.541
4.619
4.903
4.985
5.254
5.339
5.591
5.677
5.911
5.998
6.211
6.297
6.487
6.571
6.737
6.817
6.960
7.033
7.151
7.217
7.311
7.367
7.436
7.480
7.527
7.557
7.595
7.582
12.725

$$\omega^2(q) = \frac{2k}{m}\{1-\cos q\}$$

$\{\ \omega^2(q) = \omega^2(-q)$

$\Rightarrow$ degeneracy

The degeneracy of eigenvalues of the uniform system $k'=k$ is slightly broken by $k' \neq k$

defect mode has high frequency which is split off from all others!

N=64 K=1.9 Kp=1.2

mode	norm	P
1	1.000000	64.000000
2	0.998540	43.960378
3	0.999291	43.114207
4	1.001196	41.932611
5	1.000070	49.878165
6	1.000916	42.626678
7	1.000916	42.626678
8	0.999596	43.476302
9	0.999720	42.641472
10	1.000916	42.626678
11	1.000916	42.626678
12	0.999304	42.754272
13	0.999720	42.641472
14	0.996768	42.917952
15	0.998734	43.148668
16	0.999664	43.534606
17	1.000916	42.626678
18	0.999720	42.641472
19	0.999732	43.094808
20	0.999356	42.963791
21	0.999410	43.617515
22	0.999720	42.641472
23	0.999946	43.069199
24	0.997416	44.016233
25	1.000528	43.339096
26	0.997670	43.429923
27	1.001376	42.964258
28	1.000916	42.626678
29	1.000916	42.626678
30	0.999000	43.679582
31	0.999720	42.641472
32	0.999720	42.641472
33	1.000916	42.626678
34	1.000916	42.626678
35	0.999720	42.641472
36	0.999544	43.071792
37	1.000916	42.626678
38	0.996768	42.917952
39	0.999586	43.665366
40	0.996768	42.917952
41	1.000064	43.286496
42	0.999492	43.292068
43	0.999850	43.814250
44	1.000916	42.626678
45	1.000916	42.626678
46	0.997557	43.270736
47	1.000788	43.285222
48	1.000948	43.069706
49	1.000916	42.626678
50	0.996768	42.917952
51	0.999422	43.789104
52	1.000916	42.626678
53	0.999618	43.184925
54	0.999858	43.149841
55	0.999720	42.641472
56	0.998176	42.967544
57	0.999900	43.163890
58	1.001552	43.044253
59	1.000916	42.626678
60	1.000916	42.626678
61	0.998176	42.967544
62	0.999736	43.987129

when $k' < k$
there is no local mode!

All $p \sim N$

63	0.998882	43.431852
64	1.000000	64.000000

eigenvalues: N=64 K=1.9 Kp=1.2

0.000
0.018
0.018
0.072
0.073
0.161
0.164
0.284
0.289
0.441
0.449
0.629
0.640
0.847
0.863
1.093
1.113
1.365
1.389
1.659
1.689
1.974
2.009
2.306
2.346
2.651
2.697
3.007
3.059
3.371
3.428
3.738
3.800
4.106
4.172
4.470
4.541
4.828
4.903
5.176
5.254
5.511
5.591
5.830
5.911
6.130
6.211
6.407
6.487
6.661
6.737
6.887
6.960
7.085
7.151
7.253
7.311
7.389
7.436
7.492
7.527
7.561
7.582
7.596

Likewise $k' < k$
has no "split off" eigenvalue.
All frequencies are part of
same band $0 < \omega^2 < 7.6$