(1) Converting $\sum_k$ into $\int d^3k$. ~~A~~ Since k space points are separated by $(2\pi/L)$, each point occupies a volume $(2\pi/L)^3$. Thus a volume Ω in k space contains $\Omega/(2\pi/L)^3$ k points.

$$\sum_k 1 = \Omega/(2\pi/L)^3 = \frac{V}{(2\pi)^3}\Omega = \frac{V}{(2\pi)^3}\int d^3k$$

$\int d^3k\, 1 = \Omega$ Dimensionally correct also

(2) The density of states $N(E)$ is defined as the number of ~~Energy levels~~ states ~~$E(k)$~~ with Energy $E(k) = E$

$$N(E) = \underset{\text{spin}}{2} \underbrace{\sum_k}_{\text{sum over all } k} \underbrace{\delta(E - E(k))}_{\text{count it if } E(k)=E} = 2\frac{V}{(2\pi)^3}\int \delta(E - E(k))$$

(3) Examples. In 3-d with $E(k) = \hbar^2k^2/2m$

$$N(E) = \frac{V}{4\pi^3} 4\pi \int k^2\, \delta\left(E - \frac{\hbar^2k^2}{2m}\right)$$

aside: $\delta(ax) = 1/a\, \delta(x)$ $\qquad \delta(f(x)) = \frac{1}{|f'(x_0)|}\delta(x-x_0)$ over

Proof: $\int f(x)\,\delta(ax)\,dx = \int f\left(\frac{y}{a}\right)\delta(y)\frac{dy}{a} = \frac{1}{a}\int f\left(\frac{y}{a}\right)\delta(y)\,dy$
$y = ax$ $\qquad = 1/a\, f(0)$

DOS-2

$$\delta\left(E - \frac{\hbar^2 k^2}{2m}\right) = \delta\left[\left(\frac{\hbar^2}{2m}\right)\left(\frac{2mE}{\hbar^2} - k^2\right)\right]$$

$f(k)$

$$|f'(k)| = \frac{\hbar^2 k}{m}$$

$$\frac{\hbar^2 k_0^2}{2m} = E$$

so get

$$\delta\left(E - \frac{\hbar^2 k^2}{2m}\right) = \frac{1}{\frac{\hbar^2 k_0}{m}}\,\delta(k - k_0)$$

$$N(E) = \frac{V}{\pi^2}\,\frac{m}{\hbar^2 k_0}\,k_0^2$$

$$\sim k_0 \sim E^{1/2}$$

$$= \frac{2m}{\hbar^2}\,\delta\left[\left(\sqrt{\frac{2mE}{\hbar^2}} - k\right)\left(\sqrt{\frac{2mE}{\hbar^2}} + k\right)\right]$$

since $|k| > 0$

$$= \frac{2m}{\hbar^2}\,\delta\left(2\sqrt{\frac{2mE}{\hbar^2}}\;\delta\left(\sqrt{\frac{2mE}{\hbar^2}} - k\right)\right)$$

$$= \frac{\cancel{2}m}{\hbar^2}\,\frac{1}{\cancel{2}}\sqrt{\frac{\hbar^2}{2mE}}\;\delta\left(\sqrt{\frac{2mE}{\hbar^2}} - k\right)$$

$$= \sqrt{\frac{m}{2\hbar^2 E}}\;\delta\left(\sqrt{\frac{2mE}{\hbar^2}} - k\right)$$

$$N(E) = \frac{V}{\pi^2}\left(\frac{m}{2\hbar^2 E}\right)^{1/2}\frac{2mE}{\hbar^2} \sim E^{1/2}$$

$$\frac{V}{\pi^2}$$

Physically: ~~Phase space~~ k states of energy E lie on surface of sphere. area $\sim k^2 \sim E$

Why does $N(E)$ not increase linearly with E then?

answer is dE/dk factor. E is changing more rapidly with k out here also.

~~More precisely suppose $E(\vec{k}) = E(k)$~~ length only

$$\cancel{\delta(E - E(k)) \approx \delta\left[E - \left(E(0) + \frac{dE}{dk}k + \ldots\right)\right]} \quad [\text{over}/\to$$

$f(k) = E - E(k)$

$\frac{df}{dk} = \frac{dE}{dk}$

$\frac{1}{|dE/dk|}$

$$\cancel{= \delta\left[E - E(0) + \frac{dE}{dk}k\right]}$$

$$= \frac{1}{\left|\frac{dE}{dk}\right|_{k_0}} \cancel{\delta\left(\frac{E - E_0}{dE/dk} + k\right)} \; \delta(k - k_0)$$

This tells us an important principle - wherever the dispersion relation $E(k)$ is flat $dE/dk = 0$ $N(E)$ could have a big peak!

Such a peak is called a van Hove singularity.

In 3-d $E(k) = \hbar^2 k^2/2m$ we are "saved" from van Hove at $k=0$ by phase space factor k^2.

Van Hove singularities are important because whenever $N(E)$ is large ⇒ lots of states of that energy ⇒ big response there (energy conservation)

consider a scattering experiment...

In fact there is no way $N(E)$ is protected.

1

Last time we introduced

$$N(E) = 2 \sum_k \delta(E - E(k))$$

the density of states. Since in any scattering experiment any energy lost (inelastic) must go into some energy level of the solid, it is important to know the distribution of levels. *

We computed $N(E)$ for $E(k) = \hbar^2 k^2/2m$ in 3-d

$$N(E) = \begin{cases} \dfrac{V}{\pi^2} \dfrac{2m}{\hbar^2} \left(\dfrac{2mE}{\hbar^2}\right)^{1/2} & E > 0 \\ 0 & E < 0 \end{cases}$$

$$N(E_F) = 3N/E_F \qquad \text{units } 1/E \text{ as expected}$$

is a handy result obtained by using

$$E_F = \frac{\hbar^2 k_F^2}{2m} \qquad N = V\, k_F^3/3\pi^2$$

$$\frac{3N}{E_F} = \frac{3V\, k_F^3/3\pi^2}{\hbar^2 k_F^2/2m} = V \frac{2m}{\pi^2} \frac{k_F}{\hbar^2}$$

$$= \frac{V}{\pi^2} \frac{2m}{\hbar^2} \left(\frac{2mE_F}{\hbar^2}\right)^{1/2}$$

* In particular peaks in $N(E)$ may show up as peaks in scattering rate, gaps in $N(E)$ will show up as regions of little scattering.

(2)

We also noted that in computing $N(E)$ we got $1/|dE/dk|$ out in front from fooling around with δ-function - Places dE/dk vanishes ol have peaks in $N(E)$

Van Hove singularities

$\boxed{\text{KEY POINT}}$

Why can we ignore interactions if so many electrons around? Doesn't high density $\Rightarrow$ interactions must dominate ?!

No! KE also increases with density and faster.

$$P.E \sim \frac{1}{4\pi\epsilon_0} e^2/r \sim \frac{1}{4\pi\epsilon_0} e^2 n^{1/3}$$

typical separation

$$n = \frac{\#\ \text{particles}}{\text{Volume}} = \frac{N}{V} = \frac{1}{r^3}$$

$$\frac{V}{N} \sim r^3$$

But we just saw

$$KE \sim n^{2/3}$$

grows faster with n than PE

Wigner crystallization: A collection of e^- forms a lattice at low density where PE dominates.

Neutron star   N neutrons

$$PE \sim -\frac{Gm^2}{r} N^2 \sim -Gm^2 N^2 V^{-1/3}$$

$\uparrow$ # pairs

typical separation $\sim V^{1/3}$

$$KE \sim +N E_F = N \frac{\hbar^2}{2m}\left(3\pi^2 \frac{N}{V}\right)^{2/3}$$

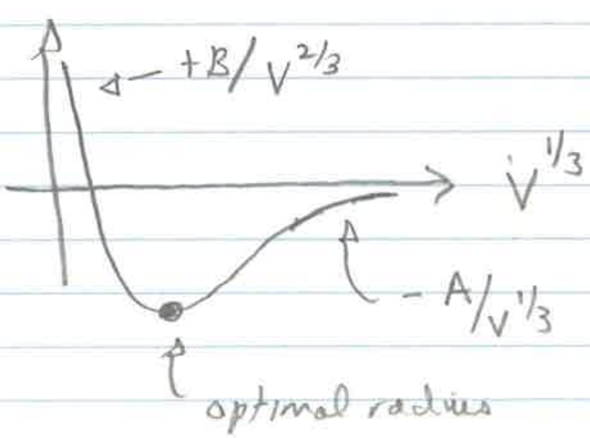
$$\sim \frac{\hbar^2}{2m}(3\pi^2)^{2/3} N^{5/3} V^{-2/3}$$

If $V \to V/8$ PE increases by $\times 2$

KE increases by $\times 4$

Balance between 2 $\to$ neutron star radius.

Need to know N, typical # of neutrons



seen before where?

8'

Neutron Star numbers

$$E = PE + KE$$

$$= -\frac{GM^2}{r} N^2 + \frac{\hbar^2}{2M} (3\pi^2)^{2/3} N^{5/3} \frac{1}{r^2}$$

$G = 6.67 \cdot 10^{-11}$

$M = 1.67 \cdot 10^{-27}$

$\hbar = 1.055 \cdot 10^{-34}$

$\rightarrow N = \frac{2 \cdot 10^{30}}{1.67 \cdot 10^{-27}}$

$M_{SUN} = 2 \cdot 10^{30}$

$$0 = \frac{dE}{dr} = \frac{GM^2N^2}{r^2} - \frac{\hbar^2}{M}(3\pi^2)^{2/3} \frac{N^{5/3}}{r^3}$$

$$r_0 = \frac{\hbar^2 (3\pi^2)^{2/3} N^{-1/3}}{GM^3}$$

$$r_0 = \frac{(1.055 \cdot 10^{-34})^2 (3\pi^2)^{2/3} \left(\frac{2}{1.67}\right)^{1/3} (10^{57})^{-1/3}}{6.67 \cdot 10^{-11} (1.67 \cdot 10^{-27})^3}$$

$$\sim 10^{-68}\, 10^{-19}\, 10^{11}\, 10^{81} \quad \frac{(1.055)^2 (3\pi^2)^{2/3} \left(\frac{2}{1.67}\right)^{1/3}}{(6.67)(1.67)^3}$$

$\uparrow$?

10^5 m $\quad$ 1/10

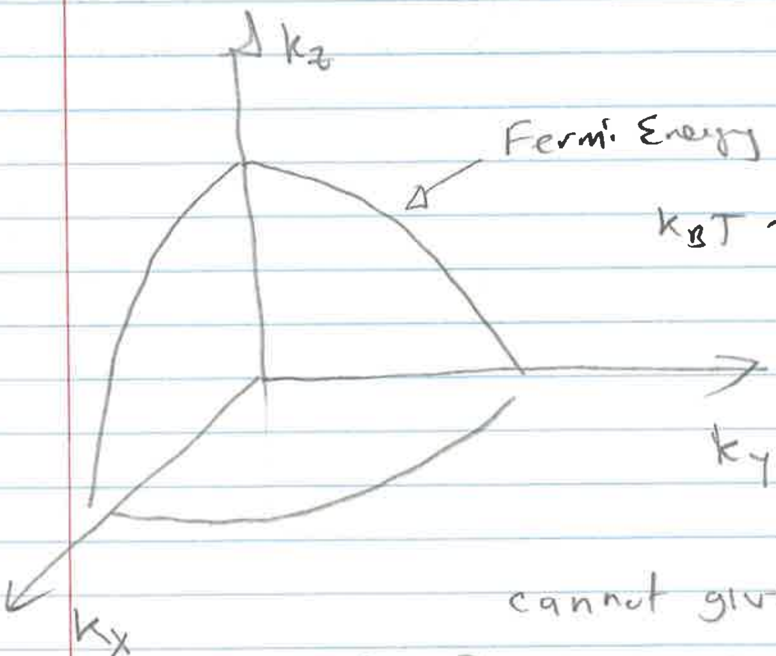
10^4 m $\sim$ 10 km

9.

Specific heat of solid

$$c(T) = \gamma T + AT^3$$

Crude argument



Fermi Energy $\sim 5 \cdot 10^{-19}$ J

$k_B T \sim 4 \cdot 10^{-21}$ J

cannot give Energy to electron (deep) inside Fermi sphere because all neighboring states filled (Pauli Blocked)

Only states within $k_B T$ of E_F can absorb energy (respond to increase in T)

$$C \sim N k_B \frac{k_B T}{E_F} \qquad \therefore \quad C \sim \gamma T$$

classical ideal gas answer

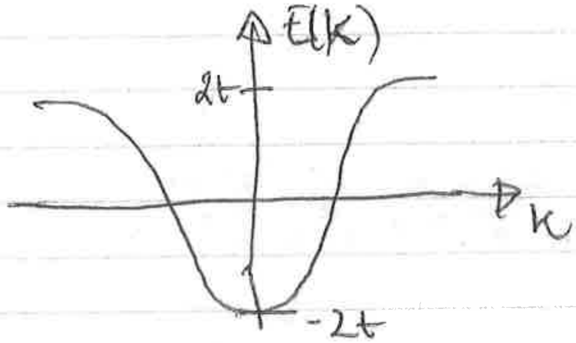
Pauli Blocking reduction

This is all at $T=0$, assume e^- have minium possible energy by occupying lowest states possible.

From our review of stat mech, as T increases, we begin to occupy higher states

E_n ⇽ can be occupied or empty but not multiply occupied!

<u>Example B</u> ~~A~~ one frequently encounters in 1-d models a dispersion relation $E(k) = -2t\cos k$



What is $N(E)$ for such a model?

1-d	dk
2-d	$k\,dk$
3-d	$k^2 dk$

In 1-d there are no phase space factors so all we worry about is where dE/dk vanishes.

→ $N(E)$ should have peaks where $E = \pm 2t$?

Explicit calculation

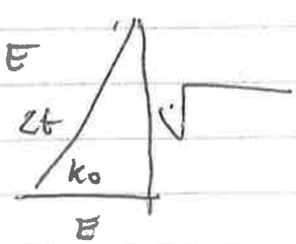
$$N(E) = 2\frac{L}{2\pi}\int dk\, \delta(E - \underbrace{2t\cos k}_{f(k)})$$

$f'(k) = 2t\sin k$

Another theorem

$$= 2\frac{L}{2\pi}\frac{1}{2t\sin k_0}$$

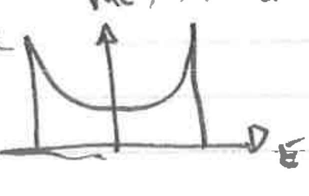
$2t\cos k_0 = E$



$$N(E) = \frac{L}{\pi}\frac{1}{\sqrt{4t^2 - E^2}}$$

$\sin k_0 = \dfrac{\sqrt{4t^2 - E^2}}{2t}$

so $N(E)$ indeed has a peak at $E = \pm 2t$.



~~Hofstee~~.

Example C D Likewise $E(k) = -2t(\cos k_x + \cos k_y)$

$- 2t(\cos k_x + \cos k_y + \cos k_z)$

in 2-dim. Hubbard model.

HW problem: compute $N(E)$ (numerically)?

~~Kuei's pictures~~

(4) Usefulness - Again where $N(E)$ is big lots of states to respond to external perturbation.

Analytically Suppose some asks you to compute any quantity that depends on k only through E even at finite temperature

$$\sum_k Q(E(k)) \frac{1}{e^{\beta(E(k)-\mu)}+1} = \frac{V}{(2\pi)^3}\int d^3k$$

$$\cancel{= \frac{V}{(2\pi)^3}\int d^3k\, Q(E) \frac{1}{e^{\beta(E-\mu)}+1}\delta(E-E(k))}$$

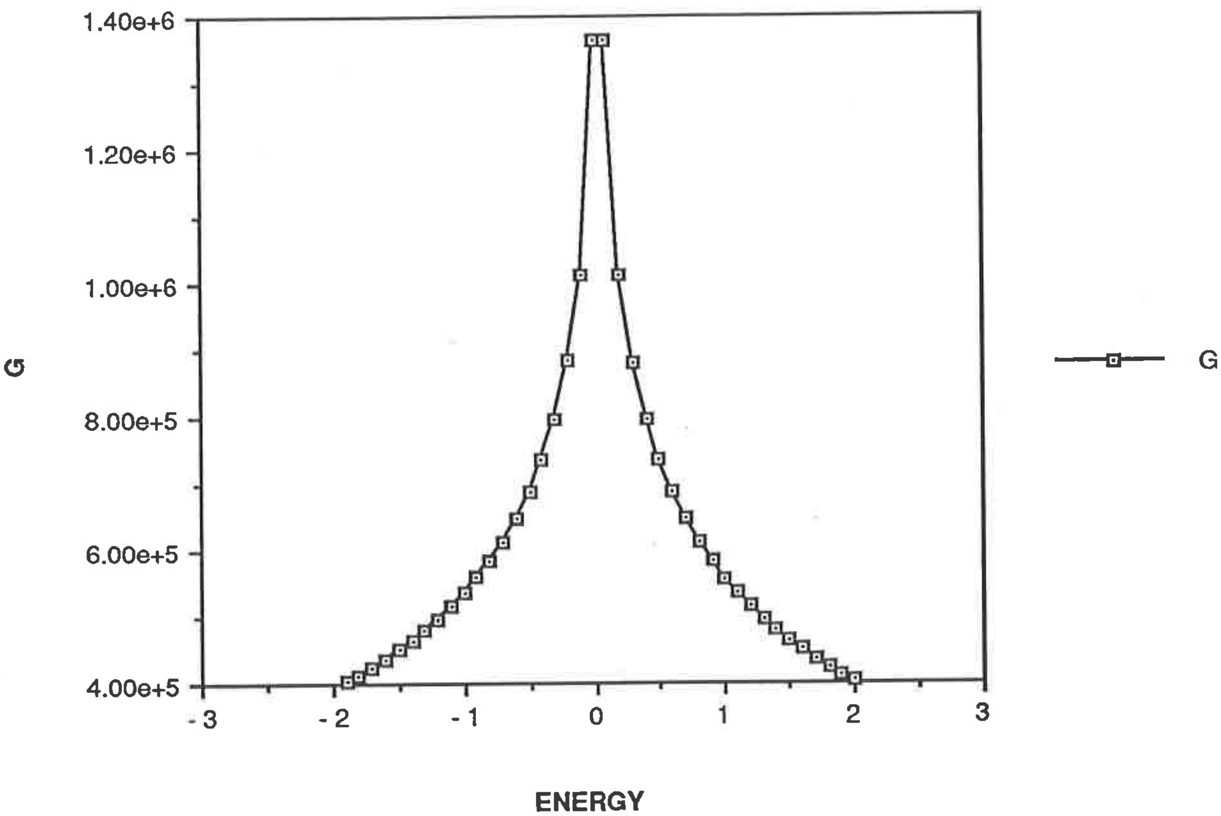
$$= \sum_k \int dE\; Q(E) \frac{1}{e^{\beta(E-\mu)}+1}\delta(E-E(k))$$

$$= \int dE\; Q(E) \frac{1}{e^{\beta(E-\mu)}+1} N(E)$$

(5) A frequent approximation $N(E) = N(E_f)$ density of states at Fermi surface. Why is that?

$$\epsilon_k = -2t(\cos k_x + \cos k_y)$$

DENSITY OF STATE (2D)



$$T_c = \omega_D \, e^{-1/\lambda N(E_f)}$$

usually $\lambda N(E_f) \sim 1/4$

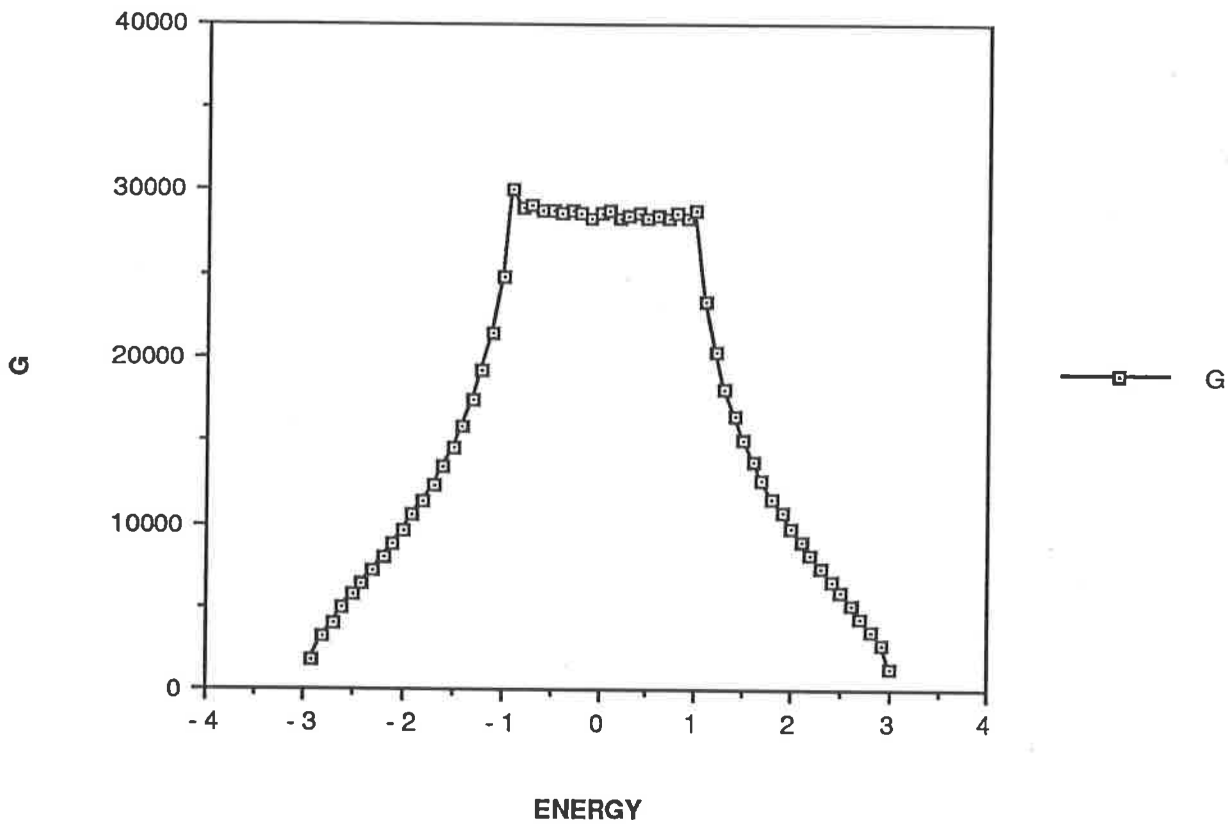
$\omega_D \sim 500\,°K$

$N(E_f)$ diverges

$500\, e^{-4} = 9.14$

$$\epsilon(k) = -2t(\cos k_x + \cos k_y + \cos k_z)$$

DENSITY OF STATE (3D)



1.

Thermodynamics : Prob of occupying E_k $\frac{1}{e^{\beta(E_k-\mu)}+1}$

$$N = 2\sum_k 1/e^{\beta(E_k-\mu)}+1 \to \frac{V}{4\pi^3}\int d^3k\ 1/e^{\beta(E_k-\mu)}+1$$

$$E = 2\sum_k E_k\ 1/e^{\beta(E_k-\mu)}+1 \to \frac{V}{4\pi^3}\int d^3k\ E_k\ 1/e^{\beta(E_k-\mu)}+1$$

We evaluated these expressions at $T=0$ where [ ] = . . .

Then we took a further step and introduced

$$N(E) = 2\sum_k \delta(E-E_k) = \frac{V}{4\pi^3}\int d^3k\ \delta(E-E_k)$$

which allowed us to rewrite

$$N = \frac{V}{4\pi^3}\int d^3k \int dE\ \delta(E-E_k)\ 1/e^{\beta(E-\mu)}+1$$

$$= \int dE\ N(E)\ 1/e^{\beta(E-\mu)}+1$$

likewise $E = \int dE\ E\,N(E)\ 1/e^{\beta(E-\mu)}+1$

$= 3N/E_F$

where $N(E) = 2Vm/\hbar^2\pi^2 \sqrt{\frac{2mE}{\hbar^2}}$ $E>0$ for $E = \frac{\hbar^2k^2}{2m}$

in 3-d

0 $E<0$

Again we know how to do calculations at $T=0$

[ ] =

$$N = \int_{-\infty}^{\mu} dE\ N(E)\ \ldots = \frac{V}{3\pi^2}\left(\frac{2m\mu}{\hbar^2}\right)^{3/2}$$

1A

$$N(E) = \frac{V}{4\pi^3} \, 4\pi \int k^2 \, dk \, \delta\left(E - \hbar^2 k^2 / 2m\right)$$

$f(k)$

$f'(k) = \frac{\hbar^2 k}{2m}$ $\qquad$ $E = \frac{\hbar^2 k_0^2}{2m}$

$$= \frac{V}{\pi^2} \frac{2m}{\hbar^2 k_0} k_0^2$$

$$= \frac{V}{\pi^2} \frac{2m}{\hbar^2} \left(\frac{2mE}{\hbar^2}\right)^{1/2}$$

Differs by factor 2V from A/M $g(E)$

$g(E_F) = \frac{3}{2} \, n/E_F$ is a convenient form. (A-M)

$N(E_F) = 3 \, N/E_F$

2.

Sommerfeld developed a technique for computing these integrals at finite T.

$$\int_{-\infty}^{\infty} H(E)\, \frac{1}{e^{\beta(E-\mu)}+1}\, dE$$

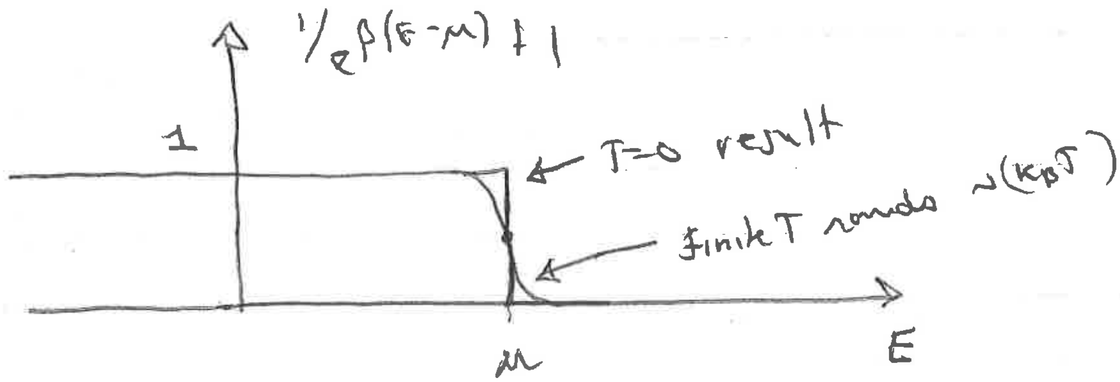
$H(E)$ is any function of E — $EN(E)$, $N(E)$, whatever

$$= \int_{-\infty}^{\mu} H(E)\, dE + \frac{\pi^2}{6}(k_B T)^2 H'(\mu) + \frac{7\pi^4}{360}(k_B T)^4 H'''(\mu) + \dots$$

Comments:

(1) Reduces to our $T=0$ result

(2) Motivation



Applications

$$N = \int_0^{\mu} N(E)\,dE + \frac{\pi^2}{6}(k_B T)^2 N'(\mu) + \cdots$$

$$E = \int_0^{\mu} E N(E)\,dE + \frac{\pi^2}{6}(k_B T)^2 \left[\mu N'(\mu) + N(\mu)\right] + \cdots$$

E_F is the dividing line between Empty & occupied states at $T=0$

μ is the quantity one must put into f_{FD} to get N right

at $T=0$ $\mu = E_F$.

As we will see $\mu = E_f + o(k_B T)^2$ so

$$\int_0^{\mu} N(E)\,dE = \underbrace{\int_0^{E_f} N(E)\,dE}_{\text{old } N} + N(E_F)(\mu - E_f)$$

Similarly for E . write it down

$$0 = N(E_F)(\mu - E_f) + \frac{\pi^2}{6}(k_B T)^2 N'(E_F)$$

$$\mu = E_F - \frac{\pi^2}{6}(k_B T)^2 N'(E_F)/N(E_F)$$

$$N(E) = cE^{1/2}$$

$$N'(E) = \tfrac{1}{2} c E^{-1/2}$$

$$N'(E)/N(E) = 1/2E$$

$$\boxed{\mu = E_F - \pi^2/6\,(k_B T)^2\, 1/2E_F} = E_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{E_F}\right)^2\right]$$

very small

Remember $E_F = 10^4 - 10^5$
$T \sim 10^2$

4

$$E = \int_0^{E_F} E N(E)\, dE + \underbrace{(\mu - E_F) E_F N(E_F)}_{\checkmark} + \frac{\pi^2}{6}(k_B T)^2 \left[E_F N'(E_F) + N(E_F) \right]$$

$\int_0^{E_F} E N(E)\, dE = E_0$

checked terms cancel by eqn for N.

$$E = E_0 + \frac{\pi^2}{6}(k_B T)^2 N(E_F)$$

$$C = dE/dT = \frac{\pi^2}{6}(k_B T)\, k_B N(E_F)$$

$$N(E_F) = 3N/E_F$$

$$\frac{C}{N} = \frac{\pi^2}{2}\left(\frac{k_B T}{E_F}\right) \cdot k_B$$

reduced by $\frac{k_B T}{E_F} \sim 10^{-3}$

classical $C = \frac{3}{2} N k_B$ no T dependence

$$\frac{C}{N} = \frac{3}{2} N k_B$$

amazing.

why?

Physics of suppression: Pauli prevents excitations of most of electron

* Prediction of linear specific heat is impt consequence of FD statistics. Actually as we shall see

$$C = \gamma T + A T^3$$

electrons phonons

usually plot $C/T = \gamma + A T^2$

We wrote

$$C = \pi^2/6 \,(k_B T)\, k_B N(E_F) \qquad N(E_f) \sim m^{3/2}$$

Comment on "heavy fermion" materials CeA_2, $CeAl_3$, ... unpaired f electrons

originally studied why

(1) No magnetic order - how come?

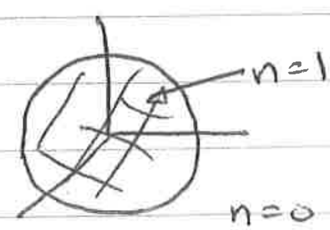
(2) ~~XX~~ γ is unusually large - Attributed to

large "m"

(3) SC.

← Page 5A →

Filled Fermi sphere



Broadened ~~at~~ surface by $k_B T$ - These are thermal excitations.

what about excitations caused by external probes.

5A E_3 — E_2 — E_1 —

STAT MECH #1

2-1930

CANONICAL PARTITION FUNCTIONS

	Classical (Distinguishable)	~~Fermion~~ Boson	~~Boson~~ Fermion
$N=0$	1	1	1
$N=1$	$e^{-\beta E_1}+e^{-\beta E_2}+e^{-\beta E_3} = \sum e^{-\beta E_i}$	$\longrightarrow$	$\longrightarrow$
$N=2$	$e^{-2\beta E_1}+e^{-2\beta E_2}+e^{-2\beta E_3} + 2e^{-\beta(E_1+E_2)} = (\sum e^{-\beta E_i})^2$	$e^{-2\beta E_1}+e^{-2\beta E_2}+e^{-2\beta E_3} + 1e^{-\beta(E_1+E_2)}$	$e^{-\beta(E_1+E_2)}+e^{-\beta(E_2+E_3)}+e^{-\beta(E_1+E_3)}$

$(\sum e^{-\beta E_i})^N$

"Smallest Number of Possibilities"

$Z_N = z^N$

$$\Omega = \sum_{N=0}^{\infty} z^N e^{-\beta\mu N} \qquad \Omega = \prod_i (1-e^{-\beta E_i})^{-1}$$

grand canonical

$$\sum_N \left[e^{-\beta E_1}+e^{-\beta E_2}+\dots\right]^N \qquad \frac{1}{1-e^{-\beta E_1}}\,\frac{1}{1-e^{-\beta E_2}}\,\frac{1}{1-e^{-\beta E_3}} \qquad \Omega = \prod_i (1+e^{-\beta E_i})$$

$= 1+\sum e^{-\beta E_i} + \dots$

$N=0$

$N=1$

$N=2$

$2e^{-\beta(E_1+E_2)}$

$(1+e^{-\beta E_1}+e^{-2\beta E_1}+\dots)$

$(1+e^{-\beta E_2}+e^{-2\beta E_2}+\dots)$

$(1+e^{-\beta E_3}+e^{-2\beta E_3}+\dots)$

or put in μ to control filling

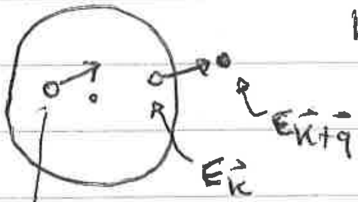
$$= 1 + e^{-\beta E_1}+e^{-\beta E_2}+e^{-\beta E_3} + 1e^{-\beta(E_1+E_2)} + \dots$$

6.

I was a bit cavalier in saying scattering $\propto N(E)$ only N_{init} no conservation laws, create phonons at will.
(1) This is not so precise. Must consider energy and momentum
(2)

More precise description of excitation $(T=0)$

"Electron-Hole Pair"



This guy cannot absorb $\vec{q}$.

Knock an electron out of filled fermi sphere.

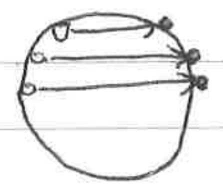
Transfer momentum $\vec{q}$. What is energy transfer

$E(k+q)-E(k)$

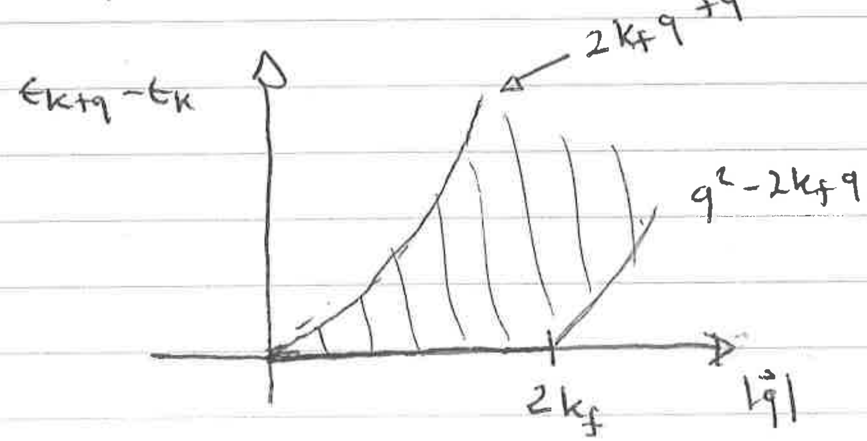
Actually for a given $\vec{q}$ not any $\vec{k}$ was legal. Must go to an unoccupied state.

For any $\vec{q}$ with $|\vec{q}| < 2k_f$ can always find a $\vec{k}$ so that $E(k+q)=E(k)$

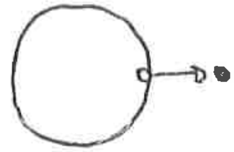
wlog choose $\vec{q} \parallel \hat{x}$



What is the "dispersion relation" the relation between momentum $\vec{q}$ transferred and energy transferred?
A misnomer!
No unique relationship
For example



7.



Suppose $\epsilon_{k+q} - \epsilon_k > 0$. Cannot do it unless $|\vec{q}|$ is sufficiently large! In fact since

$$\epsilon_{k+q} - \epsilon_k = 2\vec{k}\cdot\vec{q} + \vec{q} = 2|\vec{k}||\vec{q}|\cos\theta + q^2$$

difference is largest by making $\vec{k} \parallel \vec{q}$ and in fact making $|\vec{k}| = k_f$

$$[\epsilon_{k+q} - \epsilon_k]_{max} = 2k_f q + q^2$$

If $|\vec{q}| > 2k_f$ there is also a minimum energy you can transfer by making $\vec{k} = -k_f\hat{q}$

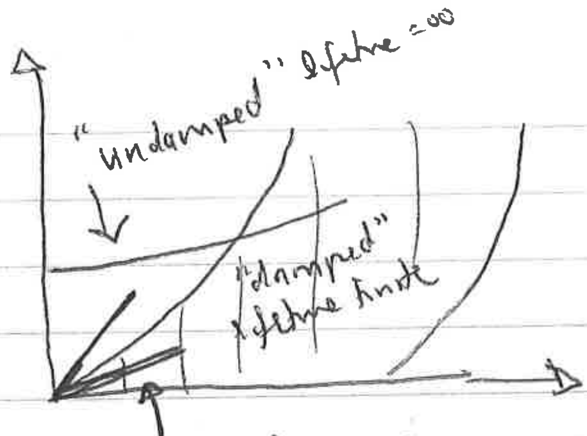
$$[\epsilon_{k+q} - \epsilon_k]_{min} = q^2 - 2k_f q$$

Draw picture as we go

"particle Hole Continuum"

Relevance Consider a probe which might transfer energy ΔE and momentum $\vec{q}$ if $\Delta E, \vec{q}$ pair lies outside shaded area cannot lose energy to excitation of el-hole pair [Maybe some other mechanism within solid]

More generally suppose there are other excitations in solid — they will be undamped



$E = \hbar \omega$

$p = \hbar K$ $\quad E = cp$

~~$\omega = ck$~~

sound waves $\omega = cq$ $\quad c \sim 10^3$ m/sec $\quad$ So sound waves are damped

$\omega = 2 v_f q$ $\quad v_f \sim 10^5$ m/sec

We saw one correction to Drude $C \sim T$ instead of $C_{classical} \sim$ constant also $c \ll c_{classical}$. Turns out κ also reduced by same factor. So Lorenz okay.

"Ultrasonic Attenuation" we will calculate this.

Hall effect ok turns out to be okay.

Perhaps not obvious, Q.M. → uncertainty principle as well as Pauli principle which required FD distribution.

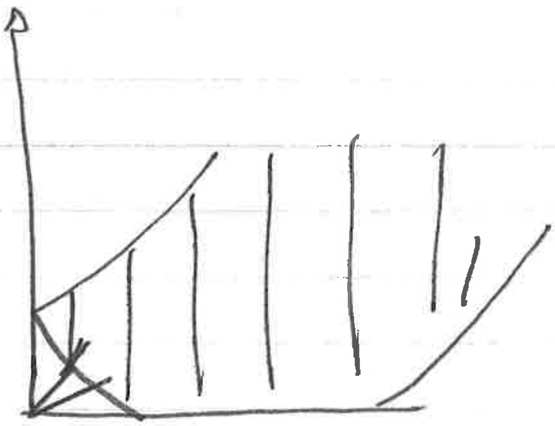
However E/M fields & Temp gradients vary over scales of $\Delta x \sim 10^3$ Å, Δp can be quite small.

Exception! X rays.

Drude also assumes $\Delta x < \ell = v_f \tau$

Fortunately $v_f \tau \approx 100$ Å also large enough so that uncertainty does not enter.

Some more examples: It turns out that if a magnetic field is applied to the sample



Now sound waves of low enough ω/q undamped

Also spin waves undamped $\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$

These effects can be seen expt'ly.