$$H_0 = -t\sum(c_\ell^\dagger c_{\ell+1} + c_{\ell+1}^\dagger c_\ell)$$

In space of $N=1$ electron basis is

$|1000\ldots\rangle$

$|0100\ldots\rangle$

Matrix of H is $\begin{pmatrix} 0 & -t & & \\ -t & 0 & -t & \\ & -t & 0 & \\ & & & \ddots \end{pmatrix}$

Eigenvalues $-2t\cos k$

What is $V = \epsilon_A \sum_{\ell\,\mathrm{odd}} c_\ell^\dagger c_\ell + \epsilon_B \sum_{\ell\,\mathrm{even}} c_\ell^\dagger c_\ell$ added

$$\begin{pmatrix} \epsilon_A & -t & & \\ -t & \epsilon_B & -t & \\ & -t & \epsilon_A & \\ & & & \ddots \end{pmatrix}$$

<u>OLD</u> Method from phonon problem

$$(H_0 + V)\psi = \lambda\psi$$

$$\psi_\ell = \begin{cases} Ae^{ik\ell} & \ell \text{ odd} \\ Be^{ik\ell} & \ell \text{ even} \end{cases}$$

KS-2

l odd $\quad \epsilon_A A e^{ikl} - tBe^{ik(l+1)} - tBe^{ik(l-1)} = \lambda A e^{ikl}$

l even $\quad \epsilon_B B e^{ikl} - tAe^{ik(l+1)} - tAe^{ik(l-1)} = \lambda B e^{ikl}$

$$\epsilon_A A - 2t\cos k\, B = \lambda A$$

$$\epsilon_B B - 2t\cos k\, A = \lambda B$$

$$\begin{vmatrix} \epsilon_A - \lambda & -2t\cos k \\ -2t\cos k & \epsilon_B - \lambda \end{vmatrix} = 0$$

$$\lambda^2 - (\epsilon_A + \epsilon_B) + \epsilon_A\epsilon_B - 4t^2\cos^2 k = 0$$

$$\lambda = \frac{1}{2}\left[(\epsilon_A + \epsilon_B) \pm \sqrt{\underbrace{(\epsilon_A + \epsilon_B)^2 - 4(\epsilon_A\epsilon_B - 4t^2\cos^2 k)}_{(\epsilon_A - \epsilon_B)^2 + 16t^2\cos^2 k}}\right]$$

or $\quad \epsilon_A = \epsilon + \Delta$

$\quad \epsilon_B = \epsilon - \Delta$

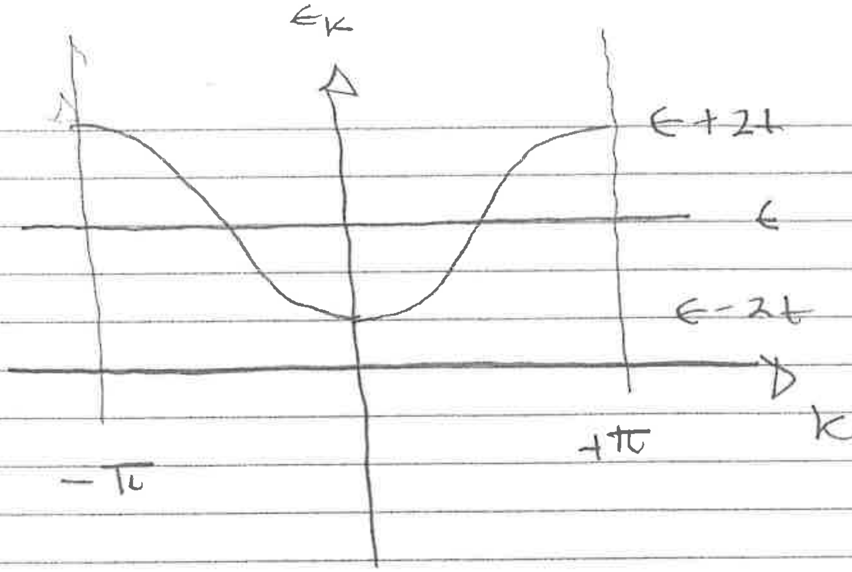
$$\epsilon_A + \epsilon_B = 2\epsilon$$

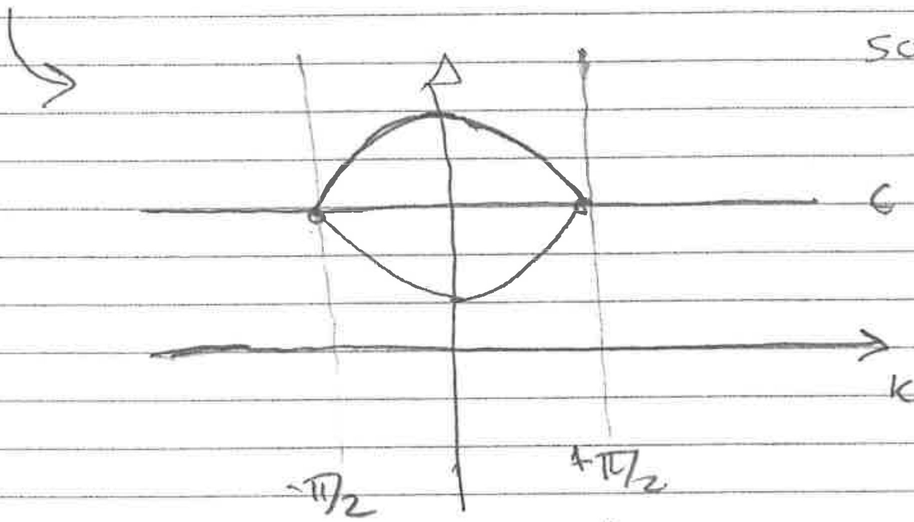
$$\epsilon_A - \epsilon_B = 2\Delta$$

$$\lambda = \epsilon \pm \sqrt{\Delta^2 + 4t^2\cos^2 k}$$

$\Delta = 0$ $\quad$ problem $\quad \lambda = \epsilon - 2t\cos k$

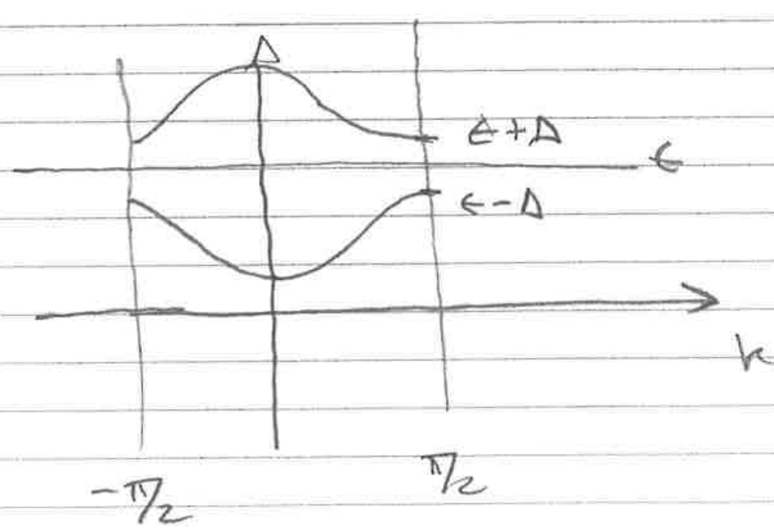
KS - 3





same list of eigenvalues

Just assign 2 ϵ_k to each of 1/2 the k point



$\epsilon \pm \sqrt{\Delta^2 + 4t^2 \cos^2 k}$

This approach is nice, but it conceals the impt physics which is that

$$V = \epsilon \sum c_\ell^\dagger c_\ell + \Delta \sum_\ell (-1)^\ell c_\ell^\dagger c_\ell$$

induces scattering between e^- of momentum k and $k+\pi$.

This scattering can be viewed as inhibiting motion of e^- which has momentum $\pi/2$ since returns it to $-\pi/2$ which precisely cancels out the motion.

KS-5

Let's do the problem in a way which better exhibits this physics!

c_l, c_l^+ destroy/create e^- at spatial site l

What's special about creating/destroying at a particular location in space? Why not create/destroy e^- of particular momentum?

$$c_k^+ = \frac{1}{\sqrt{N}} \sum_l e^{ikl} c_l^+$$

$a =$

$$c_k = \frac{1}{\sqrt{N}} \sum_l e^{-ikl} c_l$$

$k = \frac{2\pi}{N} \{1, 2, .. N\}$

or $\frac{2\pi}{N} \{-\frac{N}{2}+1, \dots \frac{N}{2}\}$

Fourier transform $f(x) = f(x+L)$

$$f(x) = \sum_{n=0}^{\infty} a_n \sin \frac{2\pi n}{L} x + b_n \cos \frac{2\pi n}{L} x$$

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{2\pi n}{L} x \, dx$$

Because

$$\int_0^L \sin \frac{2\pi n x}{L} \sin \frac{2\pi m x}{L} dx = \frac{L}{2} \delta_{nm}$$

$$\int_0^L \sin \frac{2\pi n x}{L} \cos \frac{2\pi m x}{L} dx = 0$$

sin, cos are orthogonal

KS-6

$$\int_0^L \sin\frac{2\pi m x}{L} f(x)\,dx = \sum_n a_n \frac{L}{2}\delta_{\ell m} = \frac{L}{2} a_m$$

Could also use

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi n/_L x}$$

$$\int_0^L e^{i2\pi n/_L x} e^{-i2\pi m x/_L}\,dx = \delta_{\ell m} L$$

$$c_n = \frac{1}{L}\int_0^L e^{-i2\pi n x/_L} f(x)\,dx$$

By analogy guess inversion formula

$$c_\ell^\dagger = \frac{1}{\sqrt{N}}\sum_k e^{-ik\ell} c_k^\dagger$$

$$c_\ell = \frac{1}{\sqrt{N}}\sum_k e^{ik\ell} c_k$$

Proof

$$c_k^\dagger = \frac{1}{\sqrt{N}}\sum_\ell e^{ik\ell} c_\ell^\dagger$$

$$\sum_k e^{-ikj} c_k^\dagger = \frac{1}{\sqrt{N}}\sum_\ell \underbrace{\sum_k e^{ik(\ell-j)}}_{N\delta_{\ell j}} c_\ell^\dagger$$

Discussed this before...

$$= \sqrt{N}\, c_j^\dagger$$

$$\therefore\ c_j^\dagger = \frac{1}{\sqrt{N}}\sum_k e^{-ikj} c_k^\dagger .$$

What might you worry about?

$$\{c_\ell^+, c_{\ell'}^+\} = \delta_{\ell\ell'} \iff \text{PAULI + ANTISYM}$$

$$\{c_k^+, c_{k'}^+\} = \frac{1}{N}\sum_\ell \sum_{\ell'} e^{ik\ell} e^{ik\ell'} \underbrace{\{c_\ell^+, c_{\ell'}^+\}}_{= \phi} = \phi$$

Trickier : $\{c_\ell^+, c_{\ell'}\} = \delta_{\ell\ell'}$

$$\{c_k^+, c_{k'}\} = \frac{1}{N}\sum_\ell \sum_{\ell'} e^{ik\ell} e^{-ik'\ell'} \underbrace{\{c_\ell^+, c_{\ell'}\}}_{\delta_{\ell\ell'}}$$

$$= \frac{1}{N}\sum_\ell e^{i(k-k')\ell} = \delta_{k,k'} \; !$$

$$H = -t \sum_{\ell} \left(c^{+}_{\ell} c_{\ell+1} + c^{+}_{\ell+1} c_{\ell} \right)$$

Guesses ??

$$-t \sum_{\ell} \sum_{k} \sum_{k'} \left(e^{ik\ell} e^{-ik'(\ell+1)} + e^{ik(\ell+1)} e^{-ik'\ell} \right) c^{+}_{k} c_{k'}$$

$$e^{i(k-k')\ell} \left[e^{-ik'} + e^{ik} \right] c^{+}_{k} c_{k'}$$

$\sum_{\ell}$ gives $\delta_{kk'}$

$$-t \sum_{k} 2\cos k \, c^{+}_{k} c_{k}$$

$$H = \sum -2t\cos k \, c^{+}_{k} c_{k}$$

Work in $1 e^{-}$ sector

$\# e^{-}$ of momentum k_1 etc

$|1 0 0 \ldots \rangle$

$|0 1 0 \ldots \rangle$

$|0 0 1 \ldots \rangle$

$$H = \begin{pmatrix} -2t\cos k_1 & & \\ & -2t\cos k_2 & \\ & & \ddots \end{pmatrix}$$

Now consider $V = \epsilon \sum_{\ell} c_\ell^\dagger c_\ell + \Delta \sum_{\ell} (-1)^\ell c_\ell^\dagger c_\ell$

$$\frac{1}{N} \epsilon \sum_{\ell} \sum_{k} \sum_{k'} \underbrace{e^{ik\ell} e^{-ik'\ell}}_{\delta_{kk'}} c_k^\dagger c_{k'}$$

$$= \epsilon \sum_{k} c_k^\dagger c_k$$

$\epsilon - 2t\cos k$ for $\Delta = 0$

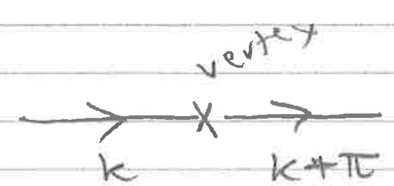
$$\Delta \sum_{\ell} (-1)^\ell c_\ell^\dagger c_\ell$$

$$\frac{1}{N} \Delta \sum_{\ell} (-1)^\ell \sum_{k} \sum_{k'} c_k^\dagger c_{k'} e^{ik\ell} e^{-ik'\ell}$$ now What ?!

$(-1)^\ell = e^{i\pi\ell}$

$$\frac{1}{N} \Delta \sum_{k} \sum_{k'} \sum_{\ell} \underbrace{e^{i\pi\ell} e^{ik\ell} e^{-ik'\ell}}_{N\,\delta_{k+\pi,k'}} c_k^\dagger c_{k'}$$

$$= \Delta \sum_{k} c_k^\dagger c_{k+\pi}$$

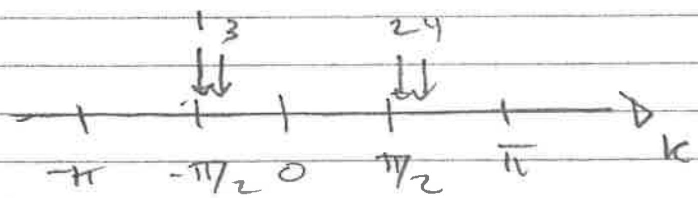


"Feynman diagram"

k and k+π mix with each other but with <u>nothing else</u>

KS-10

Arrange matrix so $k, k+\pi$ adjacent



$$\begin{pmatrix} \epsilon - 2t\cos k_1 & \Delta & \\ \Delta & \epsilon - 2t\cos(k_1+\pi) & \\ & & \end{pmatrix}$$

$$-2t\cos(k_1+\pi) = 2t\cos k_1$$

Eigenvalues $\begin{vmatrix} \epsilon - 2t\cos k_1 - \lambda & \Delta \\ \Delta & \epsilon + 2t\cos k_1 - \lambda \end{vmatrix} = 0$

$\lambda = \epsilon \pm \sqrt{\Delta^2 + 4t^2\cos^2 k}$ as in previous method!