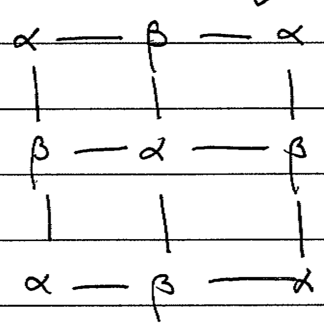
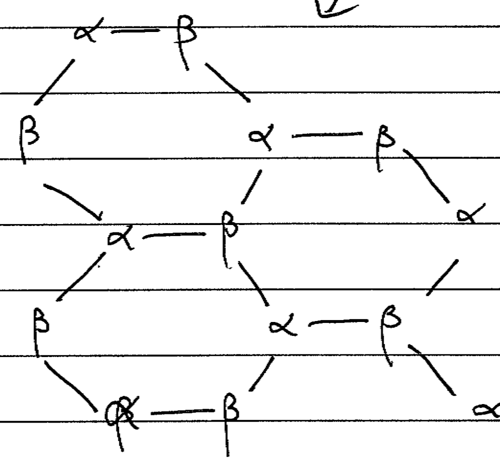
Lattices and Symmetries

A "bipartite" lattice is one in which sites divide into two classes α and β such that a site $i \in \alpha$ has only neighbors $j \in \beta$ and vice versa.

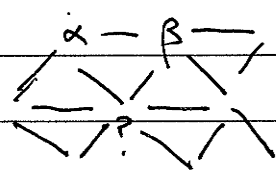
A square lattice is bipartite, as are hexagonal, cubic...





GRAPHENE

triangular is not



Stat mech has special features on bipartite lattices.

For example, the partition function Z is independent of the sign of J for the Ising model on a bipartite lattice.

L2

Ising Model

$$E = -J \sum_{\langle ij \rangle} S_i S_j \qquad S_i = \pm 1$$

Transform $S_i \to -S_i \quad \forall i \in \alpha$

Spins are still Ising (± 1 values) but

$$E = +J \sum_{\langle ij \rangle} S_i S_j$$

favors Antiferromagnetism (AF)

I am assuming $J > 0$

```
+ - + -
- + - +
+ - + -
- + - +
```

is low E configuration

Although FM is more common in everyday life (refrigerator magnets...)
AF is more common
FeO, MnO, ... ← main constituents of earth interior
La_2CuO_4, $YBa_2Cu_3O_7$ } high T_c materials
Fe-As (iron pnictides)
heavy fermions

On a bipartite lattice all of stat mech of AF is really identical to that of F! $Z(J) = Z(-J)$

"staggered magnetization" $m_s = \sum (-1)^i S_i$ ← $+1$ $i \in \alpha$, -1 $i \in \beta$

is order parameter.

Cannot draw nice AF pattern on triangular lattice!
It is "frustrated".

L3

Let's recall connection between matrix

$$D_{ij} = \begin{cases} 1 & i,j \text{ neighbors} \\ \phi & \text{otherwise} \end{cases}$$

and Laplacian operator

neighbors

$$\frac{d f}{dx^2} = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2}$$

so $\nabla^2 \rightarrow \frac{1}{(\Delta x)^2}(D - 2I)$

↑ identity matrix (neglect, just ~~trivially shifts~~ eigenvalues)

Eigenvectors of D ⟺ solns f

$D\vec{v} = \lambda \vec{v}$ $\qquad$ $\nabla^2 f = -k^2 f$

d=1

linear chain neighbor matrix ⟹

$$\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & & \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$$

Plane waves e^{ikx} $(d=1)$

$e^{i\vec{k}\cdot\vec{r}}$ $(d=3)$

$\lambda = -2\cos k \rightarrow -2(1 - k^2/2)$

$= -2 + k^2$

↑ $2I$ appearing!

For graphene $\lambda \sim k^1$ like photons!

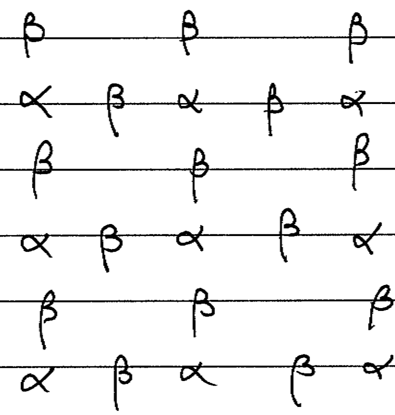
$v_\ell = e^{ik\ell}$

k discrete $\frac{2\pi}{N}\{1, .. N\}$ ← matrix dimension

Other bipartite lattices might have $N_\alpha \neq N_\beta$

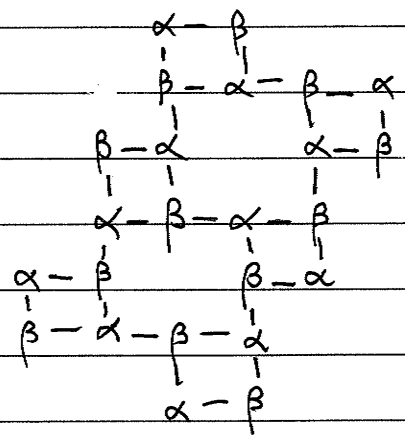
"Lieb lattice"

(aka 1/4 depleted square lattice")



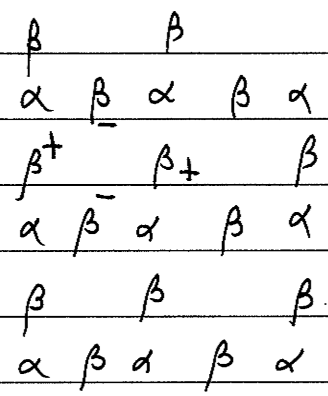
α β
$\downarrow$ $\downarrow$
CuO_2 planes of cuprate S.C.

1/5 depleted square lattice CaV_4O_9 bipartite $N_\alpha = N_\beta$



(magnetic) Vanadium atom arrangement

Lieb lattice has amazing feature: It's connectivity matrix D has an infinite # of eigenvectors with eigenvalue ϕ!



construct $\vec{v}$ which has component $v_e = 0$ $\forall e$ except ± 1 pattern around missing site

$$D\vec{v} = \phi \quad !$$

This sort of highly degenerate spectrum has many amazing implications eg topological insulators, quantum Hall Effect etc.

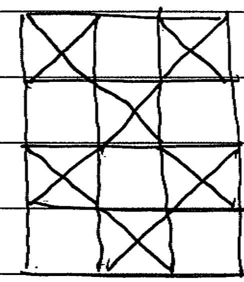
Can get lattices with this amazing infinity of $\lambda = 0$ vectors from any bipartite lattice by the following construction

① Start with bipartite lattice

② Label midpoints of each bond

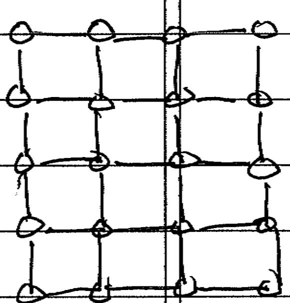
③ Connect all midpoints which share a vertex in original lattice

look familiar $J_1 J_2$ Heisenberg of 215B!

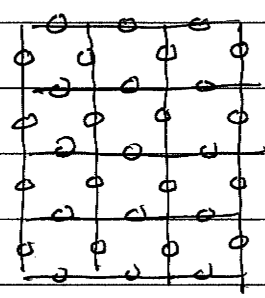


↗ redraw

Example: ① start with square lattice

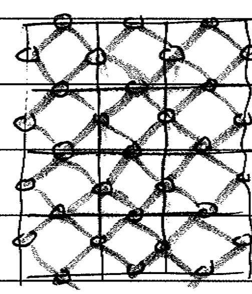


①



② label midpoints

square, smaller (lattice constant) (rotated 45°)



③ connect

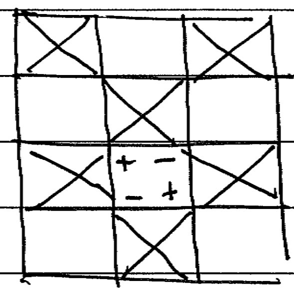
not quite square

square-plaquette

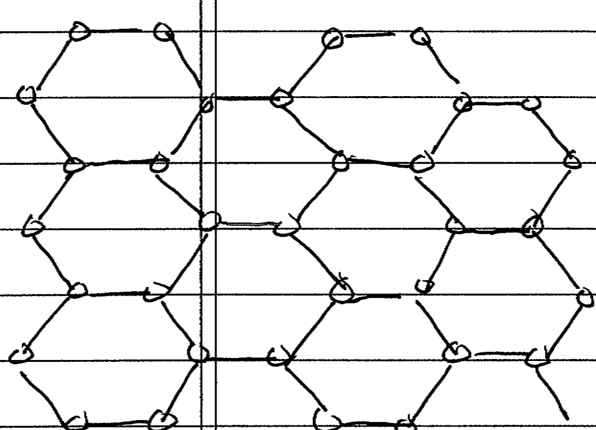
1/2 the squares have bonds across diagonals!

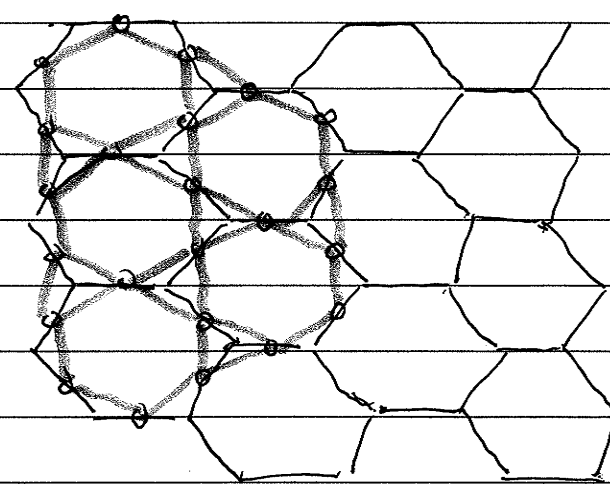
L-7

Does any one see localized states?



Example #2





"KAGOME"

chiral spin liquids!

anyons!