PROBLEM SET 5

Physics 219A, Spring 2014

Due Wednesday, May 28

[**1.**] Show that the projection operator $T_b(S'|S_1, S_2, S_3)$ for the majority rule for the three site cell discussed in class can be written explicitly in the form

$$T_b = \frac{1}{2}\Big(1 + S'\,\big(\,\lambda(S_1 + S_2 + S_3) + \mu S_1 S_2 S_3\,\big)\Big)$$

for some values of λ, μ. Use this expression to calculate the same RG equations as we derived in class, for the two cell approximation with $B = 0$. It helps if you write

$$e^{\tilde{J} S_1 S_2} = \cosh\tilde{J}\,\big(\,1 + S_1 S_2 \tanh\tilde{J}\,\big)$$

and use the same ideas as in the high T expansion.

[**2.**] Any T_b of the form above should give a RG transformation, since $\sum_{S'} T_b = 1$. In any approximation scheme, the results will depend on μ and λ. Take the case $\mu = 0$ and calculate how the fixed point $\tilde{J}_*$ depends on λ. If you adjust λ so that $\tilde{J}_*$ is at its exact value $0.275\cdots$, how well do you do on the critical exponent ν?

1-1

Physics 219A Spring 2014
PS 5 Solutions

$\boxed{1}$ A block spin transformation maps every set of spins s to another (smaller) set s'. Because each s goes to a unique s' we have

$$\sum_{s'} T_b(s|s') = 1$$

This allows us to write

$$Z = \sum_s e^{-\beta H(s)} = \sum_s \sum_{s'} T_b(s|s') e^{-\beta H(s)}$$

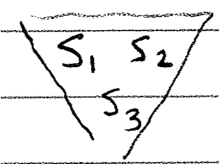
$$= \sum_{s'} \sum_s T_b(s|s') e^{-\beta H(s)}$$

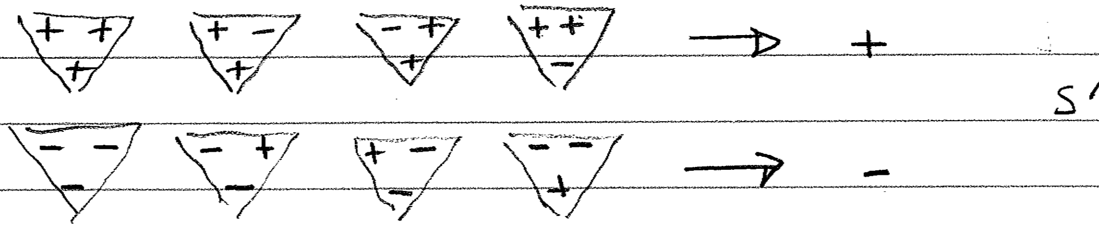
If we define $H'(s')$ by $e^{-\beta H'(s')} \equiv \sum_s T_b(s|s') e^{-\beta H(s)}$

we get $Z = Z'$. Thus our blocking leaves the partition function unchanged by determining how the new coupling constants in H' depend on the original ones in $\mathcal{H}$.

1-2

In class we considered a blocking of 3 spins on a triangular lattice via the "majority rule"





let's see this can be written as

$$T_b(S_1 S_2 S_3 | S') = \frac{1}{2}\left\{1 + S'\left(\lambda(S_1+S_2+S_3) + \mu S_1 S_2 S_3\right)\right\}$$

$$1 = T_b(1\ 1\ 1 | 1) = \frac{1}{2}\left\{1 + 3\lambda + \mu\right\}$$

$$1 = T_b(1\ 1\ {-1} | 1) = \frac{1}{2}\left\{1 + \lambda - \mu\right\}$$

It is easy to see the other 6 cases do not yield any additional eqns!

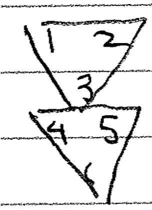
These two eqns are obeyed by $\lambda = 1/2$ $\mu = -1/2$

So we are done.

1-3

The harder task is to compute how $\tilde{J}' = \beta J'$ depends on $\tilde{J} = \beta J$. (The problem asks us only to consider the zero field case $B' = B = 0$). I will drop the "tilde" on J' and J.

The task is simple "in principle":



A $$e^{J' s_1' s_2'} = \sum_{s_1} \sum_{s_2} \cdots \sum_{s_6} T_b(s_1 s_2 s_3 | s_1')\, T_b(s_4 s_5 s_6 | s_2')\; e^{J(s_1 s_2 + s_2 s_3 + s_1 s_3 + s_3 s_4 + s_3 s_5 + s_4 s_5 + s_4 s_6 + s_5 s_6)}$$

with $T_b(s_1 s_2 s_3 | s_1') = \frac{1}{2}\left\{1 + s_1'\left(\lambda(s_1 + s_2 + s_3) + \mu s_1 s_2 s_3\right)\right\}$

and similarly for $T_b(s_4 s_5 s_6 | s_2')$

The problem is that the sum involves 64 terms and we need to do two cases $s_1' = +1$ $s_2' = +1$ and $s_1' = +1$ $s_2' = -1$ (The choices $s_1' = -1$ $s_2' = +1$ and $s_1' = -1$ $s_2' = -1$ yield no new eqns.)

1-4

Proceed as in high T expansion

$$e^{JS_1S_2} = \cosh J + S_1S_2 \sinh J = \cosh J(1+\tanh J\, S_1S_2)$$

$$t \equiv \tanh J \qquad c \equiv \cosh J$$

Consider $S_1' = S_2' = 1$

$$Ae^{J'} = \sum_{S_1}\sum_{S_2}\cdots\sum_{S_6} \frac{1}{4}\{1+\lambda(S_1+S_2+S_3)+\mu S_1S_2S_3\}$$

$$\{1+\lambda(S_4+S_5+S_6)+\mu S_4S_5S_6\}$$

$$c^8(1+tS_1S_2)(1+tS_1S_3)(1+tS_2S_3)(1+tS_3S_4)(1+tS_3S_5)(1+tS_4S_5)(1+tS_4S_6)(1+tS_5S_6)$$

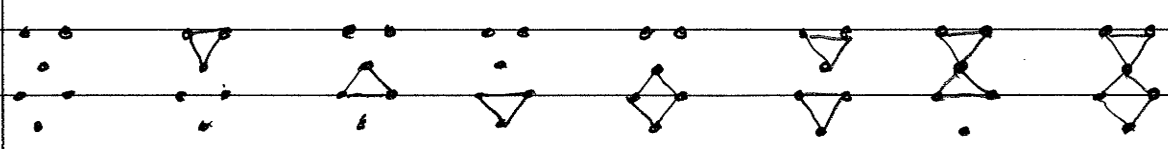
As in the high T expansion we get non zero terms

only if $\sum_{S_i}$ involves an even power of S_i

For example, if we choose the "1" from each $\{\}$

it is clear all the $(1+tS_1S_2)$ terms need to form the usual

closed loops on our 6 site cluster



$$1 \qquad \underbrace{\quad\quad}_{3t^3} \qquad t^4 \qquad \underbrace{\quad\quad}_{2t^6} \qquad t^7$$

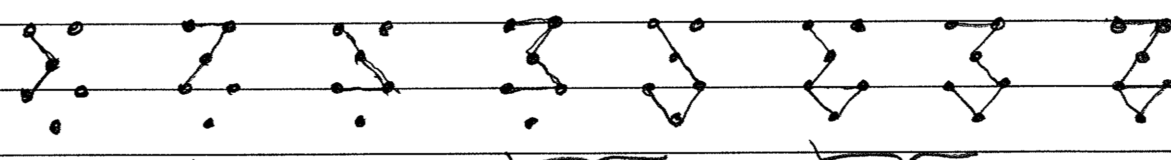
these terms contribute $(1+3t^3+t^4+2t^6+t^7)\, c^8/_4\ 2^6$

Suppose we pick λS_1 from the first $\{\}$ and 1 from the second $\{\}$

This cannot contribute because all the $(1+tS_1S_2)$ involve even #'s of spins.

We could pick λS_1 from the first $\{\}$ and λS_4 from the Second $\{\}$

Then the calculation is like going after a spin spin correlation function in the high T expansion: We need to "connect the free spins"



t^2 $\quad 2t^3 \quad 2t^4 \quad 2t^5 \quad t^6$

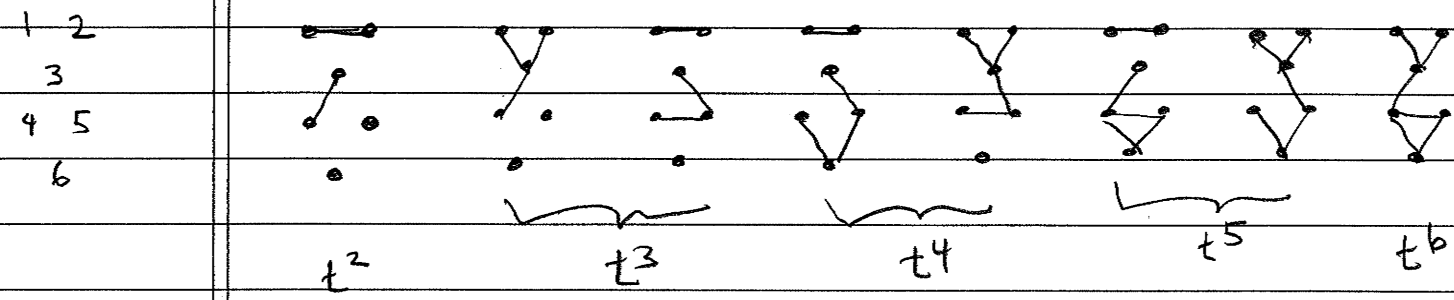
So this gives

$$\lambda^2\left(t^2+2t^3+2t^4+2t^5+t^6\right) c^8/_4\ 2^6$$

You see how it goes! Next you could do $\lambda^2 S_1 S_5 \ldots$

1-6

Let's do one that is a little different, choosing $\mu S_1 S_2 S_3$ from first {}. Clearly we must pick one of the $\lambda(S_4+S_5+S_6)$ terms from second {}. Let's do λS_4



You might want to do this in groups, either to check each other or to divide up the labor!

From this we get $\lambda\mu\left(t^2+2t^3+2t^4+2t^5+t^6\right)\frac{C^8_4}{4}2^6$

Notice by symmetry the λS_5 term must be identical.

1-5A

Completing the calculation:

1 2

3

4 5

6

We need the following:

using symmetry

$\lambda^2 S_1 S_4$ ✓ $= \lambda^2 S_2 S_5$

four more calculations

$\lambda^2 S_2 S_4 = \lambda^2 S_1 S_5$

$\lambda^2 S_3 S_4 = \lambda^2 S_3 S_5$

$\lambda^2 S_1 S_6 = \lambda^2 S_2 S_6$

$\lambda^2 S_3 S_6$

$\lambda\mu S_1 S_2 S_3 S_4$ ✓ $= \lambda\mu S_1 S_2 S_3 S_5$

3 more calculations

$\lambda\mu S_1 S_2 S_3 S_6$

$\lambda\mu S_4 S_5 S_6 S_3$

$\lambda\mu S_4 S_5 S_6 S_1 = \lambda\mu S_4 S_5 S_6 S_2$

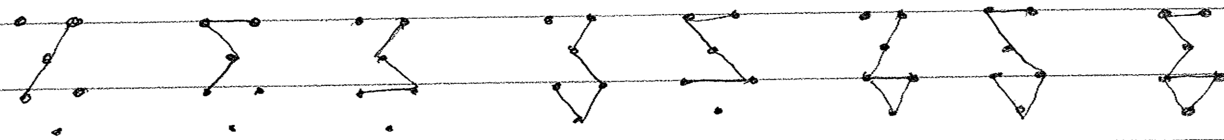
1 more calculation

$\mu^2 S_1 S_2 S_3 S_4 S_5 S_6$

1-6B

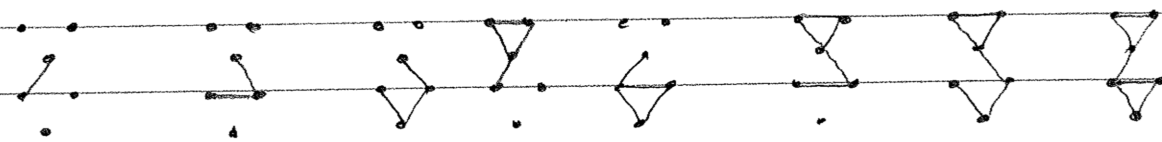
1 2
3
4 5
6

$\lambda^2 S_2 S_4$
$\lambda^2 S_1 S_5$



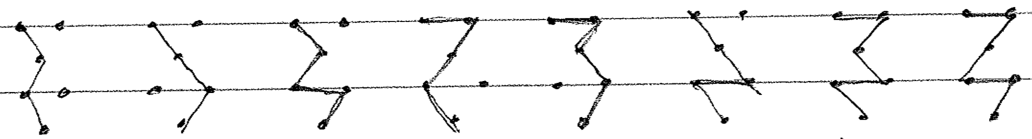
t^2 $2t^3$ $2t^4$ $2t^5$ t^6

$\lambda^2 S_3 S_4$
$\lambda^2 S_3 S_5$



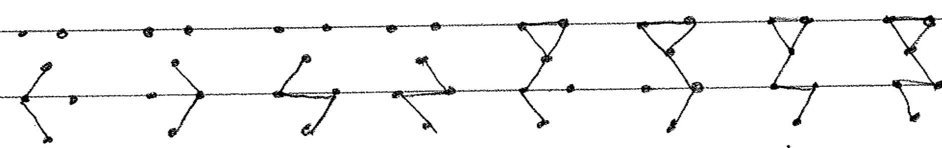
t t^2 t^3 $2t^4$ t^5 t^6 t^7

$\lambda^2 S_1 S_6$
$\lambda^2 S_2 S_6$



$2t^3$ $4t^4$ $2t^5$

$\lambda^2 S_3 S_6$



$2t^2$ $2t^3$ $2t^5$ $2t^6$

Collecting the λ^2 terms:

$$\lambda^2 \frac{c^8}{4} 2^6 \left\{ 2t + 8t^2 + 16t^3 + 20t^4 + 16t^5 + 8t^6 + 2t^7 \right\}$$

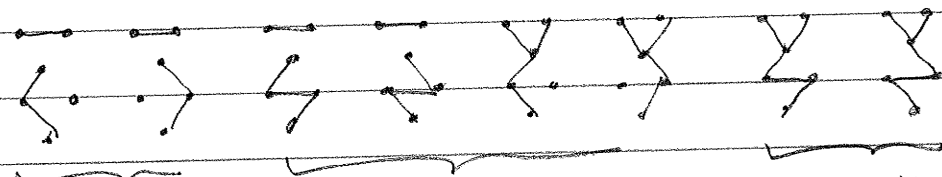
Indep of $\lambda\mu$ terms (top p1-5):

$$\frac{c^8}{4} 2^6 \left\{ 1 + 3t^3 + t^4 + 2t^6 + t^7 \right\}$$

1-bc

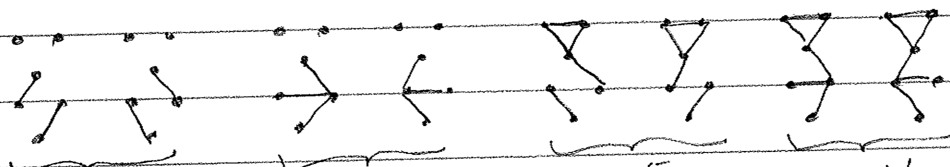
1 2
3
4 5
6

$\lambda\mu s_1 s_2 s_3 s_6$



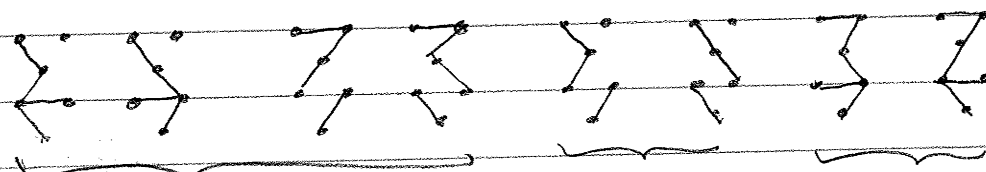
$2t^3$ $4t^4$ $2t^5$

$\lambda\mu s_4 s_5 s_1 s_3$



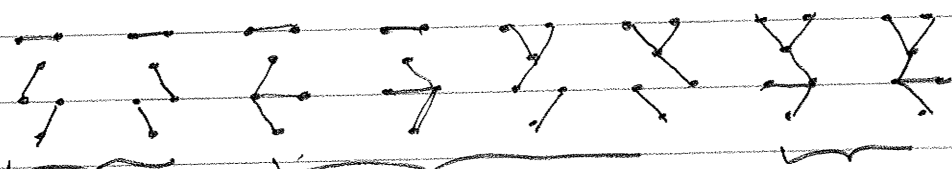
$2t^2$ $2t^3$ $2t^5$ $2t^6$

$\lambda\mu s_4 s_5 s_6 s_1$
$= \lambda\mu s_4 s_5 s_6 s_2$



$4t^4$ $2t^3$ $2t^5$

$\mu^2 s_1 s_2 s_3 s_4 s_5 s_6$



$2t^3$ $4t^4$ $2t^5$

All the $\lambda\mu$ terms:

$$\lambda\mu \frac{c^8}{4} 2^6 \left\{ 4t^2 + 14t^3 + 20t^4 + 14t^5 + 4t^6 \right\}$$

Putting Everything together for the $S_1' = S_2' = 1$ case

$$Ae^{J'} = \frac{c^8}{4} 2^6 \{ 1 + 3t^3 + t^4 + 2t^6 + t^7$$

$$+ \lambda^2 (2t + 8t^2 + 16t^3 + 20t^4 + 16t^5 + 8t^6 + 2t^7)$$

$$+ \lambda\mu (4t^2 + 14t^3 + 20t^4 + 14t^5 + 4t^6) \}$$

Now see what happens for $S_1' = 1 \quad S_2' = -1$

The difference is clearly just a flipped sign in the λ^2

and $\lambda\mu$ terms.

Setting $\lambda = 1/2 \quad \mu = -1/2$ yields, for λ^2, $\lambda\mu$ terms

$$\frac{1}{4} (2t + 4t^2 + 2t^3 + 2t^5 + 4t^6 + 2t^7)$$

$$\frac{t}{2} (1 + 2t + t^2 + t^4 + 2t^5 + t^6)$$

Notice if $\lambda = +\mu$ we would get

$$2\lambda^2 t (1 + 6t + 15t^2 + 20t^3 + 15t^4 + 6t^5 + t^6)$$

$$= 2\lambda^2 t (1+t)^6$$ very simple!

Does this suggest we did problem right (too perfect to be coincidence?!) and/or must be more simple way to do calculation?

1-8

$$e^{2J'} = \frac{f_1(t) + \lambda^2 f_2(t) + \lambda\mu f_3(t)}{f_1(t) - \lambda^2 f_2(t) - \lambda\mu f_3(t)}$$

$$f_1 = 1 + 3t^3 + t^4 + 2t^6 + t^7$$

$$f_2 = 2t + 8t^2 + 16t^3 + 20t^4 + 16t^5 + 8t^6 + 2t^7$$

$$f_3 = 4t^2 + 14t^3 + 20t^4 + 14t^5 + 4t^6$$

Or, perhaps more simply

$$Ae^{J'} = c^8/_4\, 2^6 \{ f_1 + \lambda^2 f_2 + \lambda\mu f_3 \}$$

$$Ae^{-J'} = c^8/_4\, 2^6 \{ f_1 - \lambda^2 f_2 - \lambda\mu f_3 \}$$

$$\rightarrow \quad t' = \frac{e^{J'} - e^{-J'}}{e^{J'} + e^{-J'}} = \frac{2\lambda^2 f_2 + 2\lambda\mu f_3}{2 f_1}$$

$$t' = \frac{\lambda^2 f_2 + \lambda\mu f_3}{f_1}$$

2-1

We take the eqn for J' in terms of J (actually t' in terms of t) in problem 1 and set $\mu=0$, we look for its fixed points $t'=t$.

We adjust λ to demand that the fixed point gives the exact J_c for the triangular lattice

$$t = \tanh(0.275) = 0.268$$

Then we compute $\nu = \dfrac{\ln b}{\ln\left(\frac{\partial J'}{\partial J}\right)_{J_c}}$ $\sqrt{3}$

which involves some more algebra, namely differentiation of the expression for $J'(J)$

$$\frac{dt'}{dt} = \frac{dt'}{dJ'}\frac{dJ}{dt}\frac{dJ'}{dJ} = \frac{\operatorname{sech}^2 J'}{\operatorname{sech}^2 J}\frac{dJ'}{dJ}$$

$= 1$ at $J'=J=J_c$

So we can get $dJ'/dJ\big|_{J_c} = dt'/dt\big|_{t_c}$

For $\mu = 0$ the eqn on p 1-8 becomes

$$t' = \lambda^2 f_2 / f_1$$

We want the value of λ which gives $t' = t = 0.268$

This yields $\lambda^2 = \frac{t f_1}{f_2} =$

or $\lambda = .4291$

from attached code!

Again, the final step is to get dt'/dt for this λ

and for this t. Then

$$\nu = \ln 3^{1/2} / \ln dt'/dt$$

I am not filling in details because I am not absolutely sure my factors in problem #1 are all correct. So I used a code (attached) to do it.

Numerical $dJ'/dJ \big|_{J_c = .275}$ for $\lambda = .4291$

$dJ = .005$

$J = .27500 \quad J' = .27501$

$J = .28000 \quad J' = .28436$

$dJ' = .00935$

$dJ'/dJ = 1.87$

$\nu = \ln 3^{1/2} / \ln(1.87) = .877$

exact $\nu = 1$

```
c         CODE TO DO RG FLOW FOR TRIANGULAR LATTICE ISING
c         (PROBLEM SET 5)
c         [x] WRITTEN
c         [x] COMPILED
c         [x] CHECKS AGAINST MAJORITY RULE CALCULATION
c             FOR lambda = 1/2 mu = -1/2
c                J      rgtri1.f   rgtriangular.f
c               0.1     0.061355     0.061355
c               0.5     0.593880     0.593880
c         [!] ANALYTIC FORMULA HAS (SMALL) DIFFERENCE WITH NUMERIC
c             PROBABLY MISSING SOME TERM IN EXPANSION.

c         DECLARE VARIABLES

          implicit none
          integer S1p,S2p
          real*8 lambda,mu,betaJ,betaJp
          real*8 AexpbetaJp,AexpmbetaJp
          common/tbparams/lambda,mu
          real*8 AAexpbetaJp,AAexpmbetaJp,c,t

c         INPUTS

          write (6,*) 'enter lambda,mu,betaJ'
          read  (5,*)         lambda,mu,betaJ

c         CONSIDER THE TWO CASES WHERE BLOCK SPINS ARE S1p=+1 S2p=+1
c         AND S1p=+1 S2p=-1

          S1p =  1
          S2p =  1
          call doblock(S1p,S2p,betaJ,AexpbetaJp)

          S1p =  1
          S2p = -1
          call doblock(S1p,S2p,betaJ,AexpmbetaJp)

c         COMPUTE NEW COUPLING CONSTANT

          betaJp = 0.5d0 * dlog(AexpbetaJp/AexpmbetaJp)

          write (6,*)  ' sum done numberically '
          write (6,990) AexpbetaJp,AexpmbetaJp
          write (6,990) betaJ,betaJp
990       format(2f12.6)

c         NOW CHECK THE ANALYTIC CALCULATION IN THE SOLUTION SET

          c = dcosh(betaJ)
          t = dtanh(betaJ)

          AAexpbetaJp = c**8 * 2.0**6 / 4.0 *
     1        ( 1.0 + 3.0*t**3 + t**4 +  2.0*t**6 + t**7 +
     2        lambda**2 * (  2.0*t    +  8.0*t**2 + 16.0*t**3 +
     3                      20.0*t**4 + 16.0*t**5 +  8.0*t**6 +
     4                       2.0*t**7 ) +
     5        lambda*mu * (  4.0*t**2 + 14.0*t**3
     6                      20.0*t**4 + 14.0*t**5 +  4.0*t**6 ) )

          AAexpmbetaJp = c**8 * 2.0**6 / 4.0 *
     1        ( 1.0 + 3.0*t**3 + t**4 +  2.0*t**6 + t**7 -
     2        lambda**2 * (  2.0*t    +  8.0*t**2 + 16.0*t**3 +
     3                      20.0*t**4 + 16.0*t**5 +  8.0*t**6 +
```

```
     4                       2.0*t**7 ) -
     5       lambda*mu * (  4.0*t**2 + 14.0*t**3
     6                     20.0*t**4 + 14.0*t**5 +  4.0*t**6 ) )

      betaJp = 0.5d0 * dlog(AAexpbetaJp/AAexpmbetaJp)
      write (6,*)  ' sum done analytically '
      write (6,990) AAexpbetaJp,AAexpmbetaJp
      write (6,990) betaJ,betaJp


      end


c     DO THE SUM OVER ORIGINAL SPINS GIVEN BLOCK SPIN VALUES

      subroutine doblock(S1p,S2p,betaJ,AexpbetaJp)
      integer S1,S2,S3,S4,S5,S6,S1p,S2p
      real*8 lambda,mu,betaJ,betaJp
      real*8 betaE,Tb1,Tb2,AexpbetaJp
      common/tbparams/lambda,mu

      AexpbetaJp = 0.d0
      do 160 S6=-1,1,2
      do 150 S5=-1,1,2
      do 140 S4=-1,1,2

      do 130 S3=-1,1,2
      do 120 S2=-1,1,2
      do 110 S1=-1,1,2

c          COMPUTE ENERGY

           betaE = betaJ *  ( S1*S2 + S2*S3 + S1*S3 +
     1                        S4*S5 + S4*S6 + S5*S6 +
     2                        S3*S4 + S3*S5 )

c          MULTIPLY BY Tb(S|Sp) AND ACCUMULATE

           Tb1 = 0.5d0 * ( 1.d0 + S1p *
     1          ( lambda * ( S1 + S2 + S3 ) + mu * S1*S2*S3 ) )
           Tb2 = 0.5d0 * ( 1.d0 + S2p *
     1          ( lambda * ( S4 + S5 + S6 ) + mu * S4*S5*S6 ) )

           AexpbetaJp = AexpbetaJp + Tb1 * Tb2 * dexp(betaE)

110   continue
120   continue
130   continue

140   continue
150   continue
160   continue

      return
      end
```