PROBLEM SET 3  
Physics 219A, Spring 2014  
Due Wednesday, April 30

[1.] Construct the high temperature expansion for the partition function $Z$ of the 2-d Ising model on a triangular lattice. Go out to at least sixth order in $t = \tanh(\beta J)$. Evaluate the free energy $F$ and show that all terms of higher power than linear in $N$ cancel out, so that $F$ is properly extensive.

[2.] Construct the high temperature expansion for the nearest neighbor spin-spin correlation function of the 2-d Ising model on a triangular lattice.

[3.] Construct the low temperature expansion for the partition function $Z$ of the 2-d Ising model on a triangular lattice. Do the high and low temperature expansions look related, as is the case for a square lattice? Actually, the low $T$ expansion of the Ising model on a triangular lattice is related to the high $T$ expansion on a different lattice. What is that lattice?

[4.] Look up the Onsager solution of the 2-d Ising model on a square lattice, and plot the energy $E$ as a function of temperature $T$. On the same graph, plot the high and low temperature expansion results for $E(T)$.

[5.] Solve the 1-d “XY” model by the transfer matrix technique. The Hamiltonian is

$$H = -J \sum_{l} \cos(\theta_{l} - \theta_{l+1}).$$

On each site $l$ we have an angular variable $\theta_l$ which can take on any value $0 \leq \theta_l \leq 2\pi$. Unlike the Ising case, where the transfer matrix $M$ is finite dimensional, in the XY model $M$ is infinite dimensional. (But you have encountered such things already in quantum mechanics.) The eigenvalues are the solution of an appropriate integral equation.  

Hint:

$$\exp(\beta J \cos(\theta - \theta')) = \sum_{n} I_n(\beta J) \exp(in(\theta - \theta'))$$

where $I_n(x)$ are Bessel functions. (Apparently they are everywhere!)
High T Expansion for $Z$ for 2d Ising on a triangular lattice.

\[ Z = \sum_{s_1, s_2, \ldots, s_N} e^{-\beta H} = \sum_{s_1, s_2, \ldots, s_N} e^{\beta J \sum_{\langle i,j \rangle} s_i s_j} \]

\[ = \sum_{s_1, s_2, \ldots, s_N} \prod_{\langle i,j \rangle} e^{\beta J s_i s_j} \]

\[ = \sum_{s_1, s_2, \ldots, s_N} \prod_{\langle i,j \rangle} (c + q s_i s_j) \]

\[ q = \sinh \beta J \quad c = \cosh \beta J \]

as discussed in [Ref.]; expand in closed loops

\[ Z = 2^N c^{3N} \left[ 1 + 2N t^3 + \ldots \right] \]

\[ \beta \]

select all $c$ from $(c + s_i s_j q)$

Note that all bonds are counted if 3 bonds taken from each of $N$ sites.

Here are $2N$ triangles on $N$ site lattice. If for any site you choose two plaquette shown, then all lattice is tiled.
all $t^4$ terms are counted by chasing 3 quadrilaterals
for each site

$$Z = 2^N c^{3N} \left[ 1 + 2N t^3 + 2N t^4 + ... \right]$$

The $t^5$ terms look like:

Each of these has a factor $N$.

So these give $6N t^5$

Now let's do $t^6$, clearly we have the "disconnected" pieces of 2 triangles:

$$\frac{N}{2} (N-3) t^6$$

which both point up

Similarly if both point down:

$$\frac{N}{2} (N-1) t^6$$

Similarly one up and one down triangle

$$N (N-3) t^6$$

no factor of $\frac{1}{2}$.

All together so far

$$Z = 2^N c^{3N} \left[ 1 + 2N t^3 + 2N t^4 + 6N t^5 + 2(N-1) t^6 + ... \right]$$

N.B. The disconnected pieces are a bit different from the square lattice case. We will check be consistent with requirement $F \propto N$ with no $N^2$ terms.
Remaining 6 diagrams: Hexagons

Hexagons with 2 pieces missing

There are also quadrilaterals like:

There are also triangles

all together:

\[
Z = 2^N c^{3N} \left[ 1 + 2Nt^3 + 3Nt^4 + 6Nt^5 + (2N^2 + 11N)t^6 \right] + (2N^2 + 11N)t^6...
\]

check by computing \( F = F/N \)

\[
F = -kT \ln Z
\]

use \( \ln [1 + x] = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 \ldots \)

\[
F = -kTN \left\{ \ln 2 + 3 \ln c + 2Nt^3 + 3Nt^4 + 6Nt^5 + (3N + 11)t^6 \right. \\
\left. - \frac{1}{24N}(an t^3 + an t^4 + \ldots)^2 \right\}
\]
\[ f = \frac{E}{N} = -kT \left\{ \ln 2 + 3 \ln c + 2t^3 + 3t^4 + 6t^5 + (3n + 1)t^6 \right\} - \frac{2nt^6}{T} \]
Let's compute to $t^5$. The denominator is from problem #1.

$$\langle s_i s_j \rangle_{nn} = \left[ 1 + 2Nt^3 + 2Nt^4 + 6Nt^5 + \ldots \right]^{-1} \text{ [Numerate]}$$

through $o(t^3)$ is easy:

\[ t \quad 2t^2 \quad 4t^3 \]

At $t^4$ we have disconnected pieces:

\[ (N-1)t^4 \quad (N-1)t^4 \]

and connected ones:

\[ 2t^4 \quad 4t^4 \quad 4t^4 \]
$o(t^5)$ has disconnected diagrams

\[ (N-1)t^5 \quad (N-2)t^5 \quad (N-2)t^5 \quad (N-1)t^5 \]

and a bunch of connected ones

\[ 4t^5 \quad 4t^5 \quad 4t^5 \quad 4t^5 \quad 2t^5 \]

all together:

\[
\langle 5,5; \rangle_{nn} = \left[ t + 2t^2 + 4t^3 + (2N+8)t^4 + (4N+12)t^5 + \ldots \right] \\
\left[ 1 - 2Nt^3 - 2Nt^4 - 6Nt^5 - \ldots \right] \\
R\text{ (denominator)}
\]

\[
= t + 2t^2 + 4t^3 + (2N+8)t^4 + (4N+12)t^5 \\
- 2Nt^4 - 4Nt^5 - 2Nt^5 - \ldots
\]

\[
= t + 2t^2 + 4t^3 + 8t^4 + 8t^5 + \ldots
\]
Dense line

- \( \mathcal{S} \)

2-D TRIANGULAR ISING

\( T/J \)

0.8

0.6

0.4

0.2

0

2

4

6

8

10
Low T expansion, triangular lattice

Ground state 13

Energy $E_0 = \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \end{array}$

First excited state flips 1 spin $\rightarrow -$

Each broken bond costs energy $2J$. There are 6 bonds.

So energy is higher by $+12J$

$$Z = 2e^{-\beta E_0} \left\{ 1 + Ne^{-12\beta J} + \cdots \right\}$$

$E_0$ is ground state (all +) energy

The next excited state flips 2 adjacent spins and breaks 10 bonds:

For each spin $\rightarrow 6$ neighbors

$\Rightarrow$ 3 unique adjacent neighbors (since a pair double counts)
Oswald's soln is:

\[
\frac{1}{N} < E > = - J \coth \alpha \beta J \left[ 1 + \frac{1}{\pi} \left( 2 \tanh^2 \beta J - 1 \right) K(k) \right]
\]

\[
K(k) = \int_0^{\pi/2} d\gamma \left[ 1 - k^2 \sin^2 \gamma \right]^{-1/2} \quad \text{(complete elliptic integral of first kind)}
\]

\[
k = \frac{2 \sinh 2 \beta J}{(\cosh 2 \beta J)^2}
\]

What a mess! Nevertheless we can plot this along with our low and high T expansions:

\[
\frac{1}{N} < E > = - 2J \left[ \tanh \beta J + \tanh^3 \beta J + 4 \tanh^5 \beta J + \ldots \right]
\]

\[\text{high T}\]

\[
\frac{1}{N} < E > = - 2J + 8J \left[ e^{-\beta J} + 3 e^{-12 \beta J} + 9 e^{-16 \beta J} + \ldots \right]
\]

\[\text{low T}\]

I will choose \( J = 1 \) to set scale of energies in the following plots. Also \( k_B = 1 \).

*NOTE*: The important point is that the high and low T expansions even to the low orders to which we have calculated them, cover a very wide range of T in their validity. Only near \( T_c \) do they fail, and even so, one might easily make a pretty good guess in that region.
Check ansatz at high $T$:

$\beta J$ small: $k \approx 1/\beta J$ is also small

$$I(k) = \int_{\phi}^{\pi/2} d\phi \left[ 1 - k^2 \sin^2 \phi \right]^{-1/2}$$

$$\approx \int_{\phi}^{\pi/2} d\phi \left[ 1 + \frac{1}{8} k^2 \sin^2 \phi - \cdots \right]$$

$$= \frac{\pi}{2} + \frac{\pi}{8} k^2 = \frac{\pi}{2} \left[ 1 + \frac{1}{4} \beta^2 J^2 \right]$$

$$= \sqrt{2} \left( 1 + 4 \beta^2 J^2 \right)$$

$tanh \ 2\beta J \approx 2\beta J \quad \text{and} \quad \text{cosh} \ 2\beta J \approx \frac{1}{2} \beta J$

$$\frac{1}{N} \langle \varepsilon \rangle = -J \left( \frac{1}{2\beta J} \right) \left[ 1 + \frac{1}{2\pi} \left( 8 \beta^2 J^2 - 1 \right) \right] \left( 1 + 4 \beta^2 J^2 \right)$$

$$= -J \left( \frac{1}{2\beta J} \right) \left[ 1 - (1 - 8 \beta^2 J^2) (1 + 4 \beta^2 J^2) \right]$$

$$= -J \left( \frac{1}{2\beta J} \right) \left[ 1 - (1 + 4 \beta^2 J^2) \right]$$

$$= -J \left( \frac{1}{2\beta J} \right) \left[ 4 \beta^2 J^2 \right]$$

$$= -J \left( \frac{1}{2\beta J} \right) \left[ 2 \beta J \right]$$

$$\approx -2J \tanh \beta J \quad \text{this is the first term in the high} \ T \ \text{expansion!}
$3 - y \rightarrow y - 4$

$\langle E \rangle$

$T_c$

$T_1$

$T_2$

$T_3$

onsager

2-d ISING
\[ \frac{T}{J} \]

\[ \langle E \rangle - \langle E \rangle^{ht} \]

2-D ISING
As discussed in class,

\[ Z = \int d\theta, d\theta' \, e^{iJ \cos(\theta - \theta')} e^{iJ \cos(\theta_2 - \theta_2')} \cdots e^{iJ \cos(\theta_N - \theta_N')} \]

\[ = \text{Tr } M^N \]

where \( M(\theta, \theta') = e^{iJ \cos(\theta - \theta')} \)

is the transfer matrix.

To compute its eigenvalues, note

\[ e^{iJ \cos(\theta - \theta')} = \sum_{n=-\infty}^{\infty} I_n(J) e^{i n \theta} e^{-i n \theta'} \]

\( I_n(J) \) is the Bessel function.

So \( \phi_\ell(\theta) = e^{i \ell \theta} \) is the eigenfunction:

\[ \int d\theta' M(\theta, \theta') \phi_\ell(\theta') = \int d\theta' \sum_{n=-\infty}^{\infty} I_n(J) e^{i n \theta} e^{-i n \theta'} e^{i \ell \theta} \]

\[ = \sum_{n=-\infty}^{\infty} I_n(J) e^{i n \theta} 2 \pi \delta_{\ell n} \]

\[ = 2 \pi I_\ell(J) \cdot e^{i \ell \theta} = \lambda \phi_\ell(\theta) \]

with eigenvalue \( \lambda = 2 \pi I_\ell(J) \).

For a finite chain \( Z = \sum_{n=-a}^{a} e^{iN} \) In the thermodynamic limit \((N \to \infty)\) we just need \( \Lambda \), in this case it's \( I_0 \), so \( Z = \lambda^N \).