MIDTERM 2

Physics 219A, Spring 2014

Do four of five problems!

[**1.**] Compute, by explicit enumeration of all the states, the partition function $Z(T)$ and average energy $\langle E \rangle(T)$ of the $N = 6$ site one dimensional Ising model with periodic boundary conditions. Verify your expression for Z by checking against the exact solution $Z = (2\cosh\beta J)^N + (2\sinh\beta J)^N$. If you find $N = 4$ too hard to do, try $N = 4$.

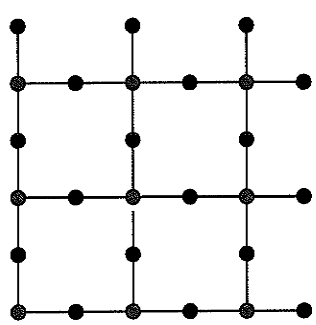
[**2.**] Make qualitative plots of the susceptibility $\chi(T)$, and the magnetization $m(T)$ at field $B = 0$ for a system with a second order magnetic phase transition. How are the critical exponents β and γ defined? Make qualitative plots of the magnetization $m(B)$ for $T > T_c$, for $T = T_c$ and for $T < T_c$. How is the critical exponent δ defined? What are the Mean Field values of β, γ, and δ? Discuss any relationships between the plots that occur to you. For example, does the behavior of $\chi(T)$ near T_c seem plausible in terms of what happens to $m(T)$?

[**3.**] Write down the high and low T expansions for the partition function of the Ising model on a hexagonal lattice. For high T go to order ten in $\tanh\beta J$. For low T go to order four in $\exp(-2\beta J)$.

[**4.**] Landau has a nice way of picturing any phase transition. What is it?

[**5.**] Suppose someone challenges you to work out Mean Field Theory (MFT) for the Ising model on the "Lieb Lattice". See below. Unlike the square lattice we did in class, where all the sites are identical, there are two different types of sites here (one type has two neighbors and the other has four neighbors). How might you generalize MFT to such a situation? How would you formulate self-consistency? How would you go about determining T_c?

<u>Hint:</u> Allow $\langle S_i \rangle$ to be different depending on which type of site is being considered.



The Lieb lattice.

1-1.

Physics 219 A
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$\boxed{1}$ There are $2^6 = 64$ possible states,

but they come in groups with the same energy

$$\left.\begin{matrix} + + + + + + \\ - - - - - - \end{matrix}\right\} \quad E = -6J \qquad g = 2\cdot 1$$

$$\left.\begin{matrix} + + + + + - \\ - - - - - + \end{matrix}\right\} \quad E = -2J \qquad g = 2\cdot 6 = 12$$

and related

6 places to put the flipped spin

$$\left.\begin{matrix} + + + + - - \\ - - - - + + \end{matrix}\right\} \quad E = -2J \qquad g = 2\cdot 6 = 12$$

and related

6 places to put the flipped pair

$$\left.\begin{matrix} + + + - + - \\ - - - + - + \end{matrix}\right\} \quad E = +2J \qquad g = 2\cdot \frac{6\cdot 3}{2} = 18$$

and related

6 places to put first flipped spin, only 3 places for second (not allowed to be adjacent) divide by 2 because indistinguishable

1-2.

same!

++−−+−
−−++−+

$E = +2J$ $\quad g = 6 \cdot 2 = 12$

6 places to put pair of flipped spins, 2 places to put single flipped spin. These are distinguishable

note: Do not need to consider −−−−−− background separately

same!

+++−−−
−−−+++

$E = -2J$ $\quad g = 6$

6 places to put +++
notice in this case do not need to consider −−−−−− background separately.

+−+−+−
−+−+−+

$E = +6J$ $\quad g = 2$

Check counting by noting $2+12+12+18+12+6+2 = 64$

Another check: Expect $E \to -E$ symmetry

Indeed there are $12+12+6 = 30$ $\quad E = -2J$

$18+12 = 30$ $\quad E = +2J$

1-3.

$$Z = 2e^{6\beta J} + 30e^{2\beta J} + 30e^{-2\beta J} + 2e^{-6\beta J}$$

$$E = -\frac{\partial}{\partial\beta}\ln Z$$

$$= J Z^{-1}\left\{12e^{6\beta J} + 60e^{2\beta J} - 60e^{-2\beta J} - 12e^{-6\beta J}\right\}$$

Or, if you like

$$Z = 4\cosh 6\beta J + 60\cosh 2\beta J$$

$$\langle E\rangle = Z^{-1} J\left\{24\sinh 6\beta J + 120\sinh 2\beta J\right\}$$

We could check against transfer matrix / high T expansion

$$Z = (2\cosh\beta J)^6 + (2\sinh\beta J)^6$$

$$= (e^{\beta J} + e^{-\beta J})^6 + (e^{\beta J} - e^{-\beta J})^6$$

$$= e^{6\beta J} + 6e^{4\beta J} + 15e^{2\beta J} + 20 + 15e^{-2\beta J} + 6e^{-4\beta J} + e^{-6\beta J}$$

$$+ e^{6\beta J} - 6e^{4\beta J} + 15e^{2\beta J} - 20 + 15e^{-2\beta J} - 6e^{-4\beta J} + e^{-6\beta J}$$

$$= 2e^{6\beta J} + 30e^{2\beta J} + 30e^{-2\beta J} + 2e^{-6\beta J} \quad \checkmark\checkmark$$

$\boxed{2}$

Fig 1

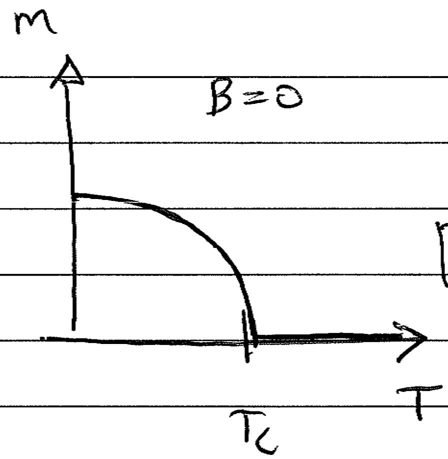
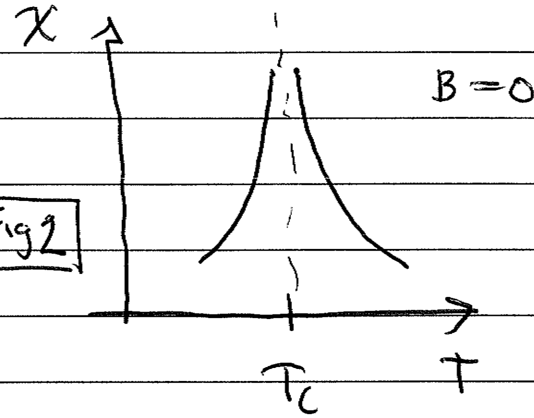


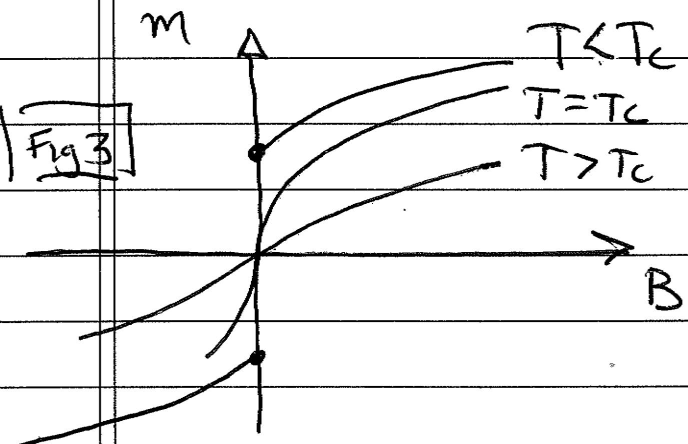
Fig 2



For T near T_c

$$m(T) \sim (T_c - T)^{\beta} \qquad \chi(T) \sim 1/(T-T_c)^{\gamma}$$

Fig 3



For $T = T_c$

$$M \sim B^{1/\delta}$$

Comments:

(1) The plot of $M(T)$ at $B=0$ tells us that the system spontaneously magnetizes if $T < T_c$. This suggests that its susceptibility to a magnetic field should be huge: it is about to line up even at $B=0$ so turning on B will have a large effect. Schematically

$$\underbrace{M}_{\text{nonzero}} = \chi \underbrace{B}_{\text{zero}} \Rightarrow \chi = \infty$$

2-2

(2) The values of $m(B=0)$ in figure 3 are consistent with those of figure 1: If $T \geq T_c$, $m(B=0)$ is zero in figure 1 and also in figure 3.
If $T < T_c$, $m(B=0)$ is nonzero in figure 1 and also in figure 3.

(3) The slope of $M(B)$ in Figure 3 is the susceptibility χ. At $T=T_c$ in figure 3, $M(B)$ comes in to $B=0$ with infinite slope (assuming $\delta > 1$). χ is also infinite in figure 2.

In MFT $\beta = 1/2$, $\gamma = 1$, $\delta = 3$

$\boxed{3}$ High T Expansion

$$Z = \mathrm{Tr}\, e^{-\beta E} = \sum_{\{S\}} e^{\beta J \sum_{\langle ij \rangle} S_i S_j}$$

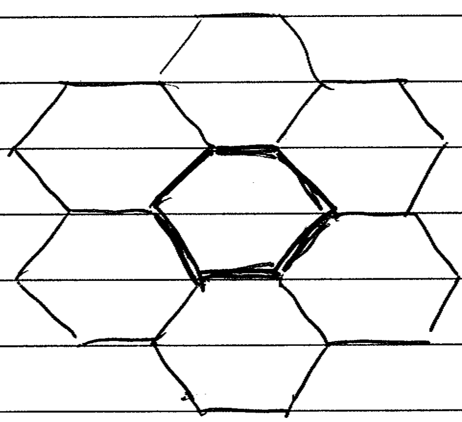
$$= \sum_{\{S\}} \prod_{\langle ij \rangle} \left(\cosh \beta J + S_i S_j \sinh \beta J \right)$$

$$= (\cosh \beta J)^{N_{bonds}} \sum_{\{S\}} \prod_{\langle ij \rangle} \left(1 + S_i S_j \tanh \beta J \right)$$

small at high T

$\sum_{\{S\}}$ gives zero unless form closed loops from any $S_i S_j \tanh \beta J$ factors.

For hexagonal lattice, smallest closed loop is



with six bonds

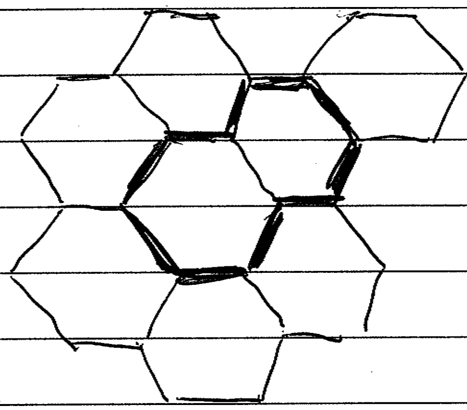
Also if N = # sites

$N_{bonds} = 3N/2$

You can only place hexagons on 1/2 as many places as # sites

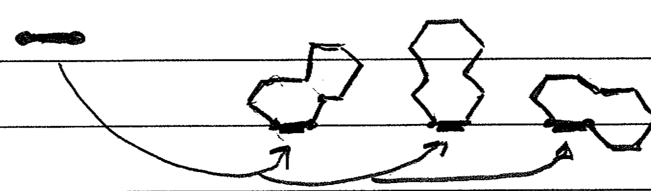
$$Z = (\cosh \beta J)^{3N/2} \left\{ 1 + \frac{N}{2} t^6 + \ldots \right\} 2^N$$

What about higher order? The next closed loop possible is this one: This is $(\tanh\beta J)^{10}$ and we need to compute number of ways it can be placed. There are $N/2$ locations but 3 distinct orientations *



$$Z = (\cosh\beta J)^{3N/2}\left\{1 + \frac{N}{2}t^6 + \frac{3N}{2}t^{10} + \dots\right\} 2^N$$

* Given a pair of sites forming a horizontal bond one can draw 3 $(\tanh\beta J)^{10}$ diagrams



low T expansion

Idea is to expand in flipped spins

Ground state E is Ferromagnetic $E = -NJ\,3/2$

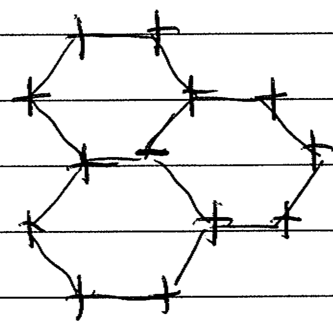
If $N = \#$ sites

$\frac{3N}{2} = \#$ bonds

$Z = 2e^{3\beta NJ/2}$

2 FM configs : all +1 or all −1

Next lowest E is



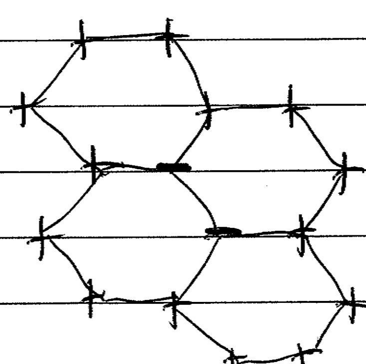
Energy cost is $6J$

($2J$ for each of 3 bonds)

Can do this on any site

$$Z = 2e^{3\beta NJ/2}\{1 + Ne^{-6\beta J} + \ldots\}$$

Or can flip 2 adjacent spins



"Breaks" 4 bonds

→ Energy cost $8J$

Can do this for any bond of lattice ⇒ $N = \#$ sites

$\frac{3N}{2} = N_{bonds}$

(again 3 bonds per site, but each bond uses 2 sites)

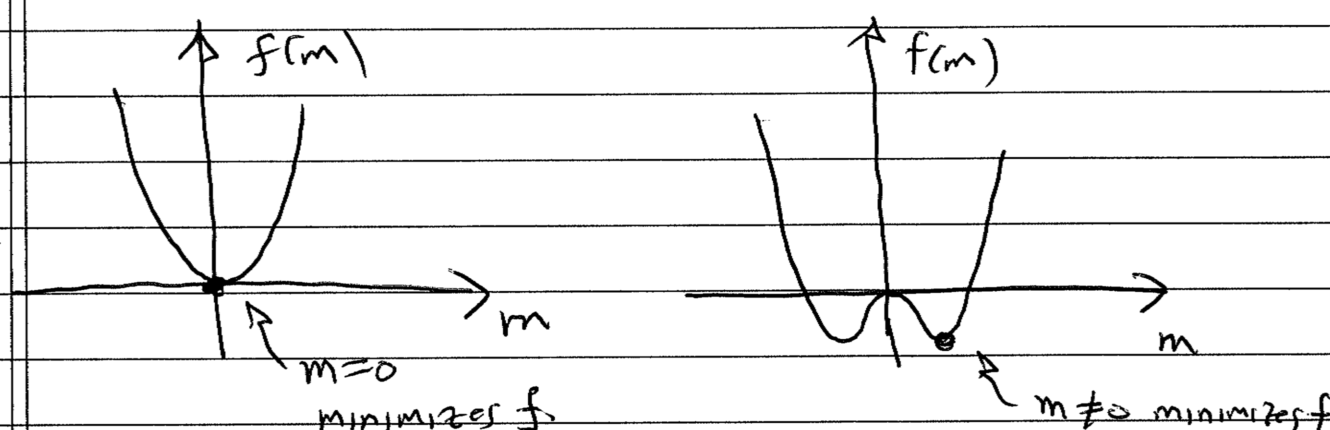
3-4

$$Z = 2e^{3\beta NJ/2}\left\{1 + Ne^{-6\beta J} + \frac{3N}{2}e^{-8\beta J} + \ldots\right\}$$

$\boxed{4}$ Landau Mean field theory pictures phase transitions in terms of the expansion of the free energy f in terms of an order parameter m

$$f(m) = f_0 + a(T)m^2 + b(T)m^4$$

for example. Assuming $b(T) > 0$ there are 2 possible shapes for $f(m)$ depending on the sign of $a(T)$



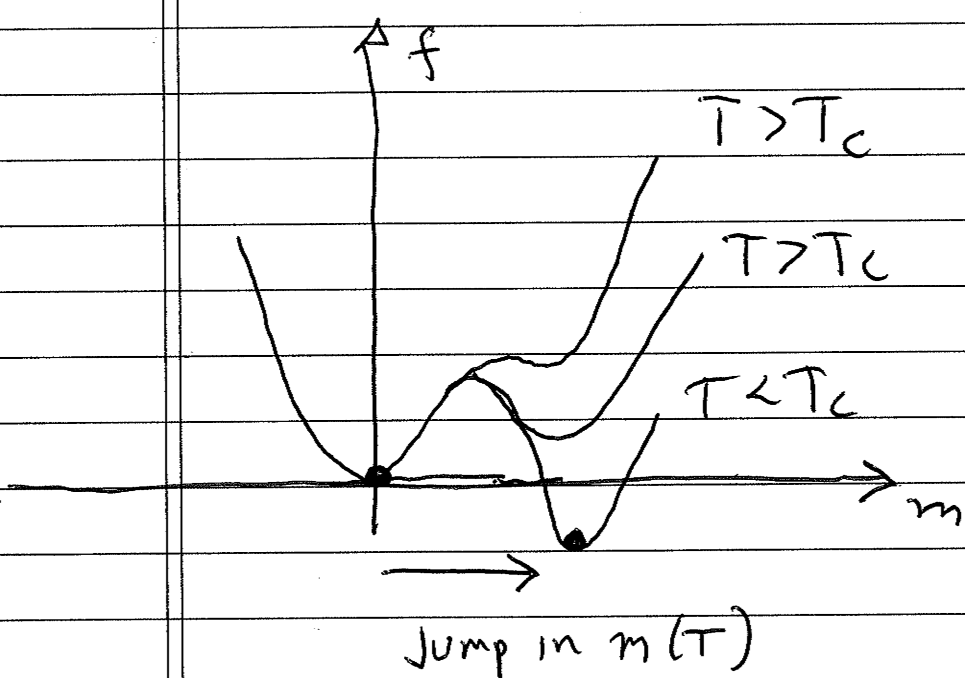
$a > 0$ $\qquad\qquad$ $a < 0$

In Landau MFT a phase transition occurs because $a(T)$ changes sign, $a(T) > 0$ $T > T_c$
$a(T) < 0$ $T < T_c$

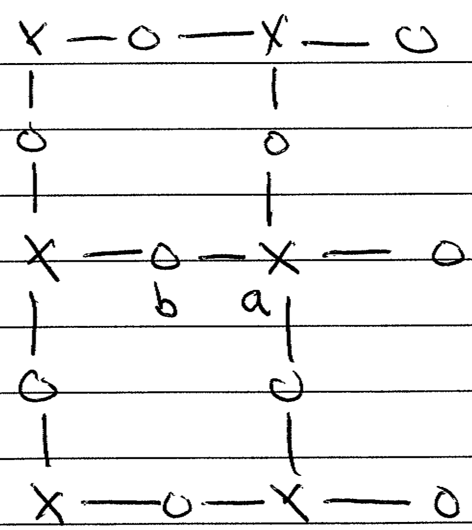
So that the order parameter $m = 0$ minimizes $f(m)$ for $T > T_c$ but $m \neq 0$ for $T < T_c$

4-2

Landau MFT also allows an understanding of first order transitions. If there is an m^3 term in $f(m)$ it is possible f can evolve with T so that the minimum jumps abruptly from $m=0$ to $m=m_0$



$\boxed{5}$ Lets denote $m_a = \langle S_i \rangle$ on the 4-fold coordinated sites and $m_b = \langle S_i \rangle$ on the 2 fold coordinated sites. MFT replaces S_j by $\langle S_j \rangle$ on the neighbors j of site i. This leads to



S_a feels $-4 J m_b$

S_b feels $-2 J m_a$

ie $E_a = -4 J m_b S_a$

$E_b = -2 J m_a S_b$

Get 2 self consistent Eqns:

$$\langle S_a \rangle = m_a = \tanh 4 J m_b \beta$$

$$\langle S_b \rangle = m_b = \tanh 2 J m_a \beta$$

an obvious solution is $m_a = m_b = 0$, and this is what happens for T large (β small).

5-2

When will there first be a solution

to these eqns with m_a and m_b nonzero?

One guess is to set $\tanh x \simeq x$

$$m_a = 4J m_b \beta$$

$$m_b = 2J m_a \beta = 2J(4J m_b \beta)\beta$$

$$= 8 J^2 \beta^2 m_b$$

So we need $\sqrt{8} J\beta = 1$

$$\Rightarrow k_B T_c = \sqrt{8} J$$

This is plausible in that it is inbetween the

values $T_c = 2J \leftarrow z=2$ lattice

$T_c = 4J \leftarrow z=4$ lattice