MIDTERM 1

Physics 219A, Spring 2014

Do only three of the four problems!

[**1.**] Consider a classical system which has three energy states $E_1 < E_2 < E_3$. Assume $E_3 >> E_2, E_1$. That is, two of the levels (E_1 and E_2) are fairly close together, while E_3 is much higher.
(a) Sketch p_1, p_2, p_3 versus temperature T.
(b) Sketch $\langle E \rangle$ versus temperature T.
(c) Sketch C (specific heat) versus temperature T.
(d) Sketch S (entropy) versus temperature T.
Comment briefly on the structure of each of your figures.

[**2.**] The ideal relativistic gas.
(a) Calculate $N(E) = \int d\Gamma \delta(E - H)$ for N free classical particles with

$$H = \sum_l c p_l.$$

(b) Find the equation of state and the specific heat.
(c) Find the single particle momentum distribution function.

[**3.**] What is the specific heat for a two level system with energies $E_1 = 0$ and $E_2 = \Delta$? One of your fellow graduate students discovers a new material with specific heat $C(T) \propto T^5$ at low temperatures, and asks for you to provide an explanation in terms of a collection of independent two level systems with some energy-spacing probability $p(\Delta) \propto \Delta^\alpha$. What do you tell her the power α would have to be? (Provide a calculation, not just a number.)

[**4.**] An atom of mass m is placed in a three dimensional cubic box with sides of length L. Treat the atom classically.
(a) What is the Hamiltonian?
(b) What is the partition function?
(c) What is $\langle E \rangle$?
(d) What is the specific heat at high and low T? Explain your answer physically.

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Midterm 1 Solutions

NB Solutions in different order than on exam!

$\boxed{1}$ $\mathcal{H} = p_x^2/2m + p_y^2/2m + p_z^2/2m + mgz$

$$Z = \int dx \int dy \int dz \int dp_x \int dp_y \int dp_z \, e^{-\beta \mathcal{H}}$$

$$= L \cdot L \cdot \int_0^L e^{-\beta mgz} dz \, (\pi 2m k_B T)^{3/2}$$

3 momentum integrals

$$= L^2 \left. \frac{e^{-\beta mgz}}{-\beta mg} \right|_0^L = \frac{1}{\beta mg} \{1 - e^{-\beta mgL}\}$$

$$Z = L^2 \left(\frac{2\pi m}{\beta}\right)^{3/2} \frac{1}{\beta mg} \{1 - e^{-\beta mgL}\}$$

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \ln Z = \frac{3}{2} k_B T + k_B T - \frac{mgL e^{-\beta mgL}}{1 - e^{-\beta mgL}}$$

Consider high T limit $\beta \to 0$ $1 - e^{-\beta mgL} \approx \beta mgL$

$$\langle E \rangle = \frac{5}{2} k_B T - mgL/\beta mgL = 3/2\, k_B T$$

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The interpretation of this result is that at really high T the atom just occupies all z values equally and the gravitational potential energy becomes irrelevant (because it is bounded by mgL) we get the usual $\frac{3}{2}k_B T$ result. Note that if $L = \infty$ we never lose the extra $k_B T$ term in $\langle E \rangle$!

$$C = \frac{d\langle E\rangle}{dT} = \frac{5}{2}k_B + k_B \beta^2 \frac{d}{d\beta} \frac{e^{-\beta mgL}\, mgL}{1 - e^{-\beta mgL}}$$

$$\beta = \frac{1}{k_B T}$$

$$\frac{d}{dT} = \frac{d\beta}{dT}\frac{d}{d\beta} = -\frac{1}{k_B T^2}\frac{d}{d\beta}$$

$$\frac{d}{d\beta} \frac{mgL}{e^{\beta mgL} - 1}$$

$$- \frac{(mgL)^2 e^{\beta mgL}}{(e^{\beta mgL} - 1)^2}$$

$$C = k_B \left\{ \frac{5}{2} - \left(\frac{mgL}{k_B T}\right)^2 \frac{e^{\beta mgL}}{(e^{\beta mgL} - 1)^2} \right\}$$

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as $T \to 0$ $\beta \to \infty$ and $\frac{e^{\beta m L g}}{(e^{\beta m g L}-1)^2}(\beta m g L)^2 \to \emptyset$

so $C \to 5/2\, k_B T$

as $T \to \infty$ $\beta \to 0$

$$(\beta m g L)^2 \frac{e^{\beta m g L}}{(e^{\beta m g L}-1)^2} \to 1$$

So $C \to 3/2\, k_B$ as expected since $\langle E \rangle \to 3/2\, k_B T$

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$\boxed{2.}$ $p(\Delta) = A\Delta^{\alpha}$ $\quad 0 < \Delta < \Delta_m$

Normalyse $1 = \int_0^{\Delta_m} p(\Delta)\, d\Delta = A \dfrac{\Delta^{\alpha+1}}{\alpha+1}\Big|_0^{\Delta_m}$

Hence $A = \alpha+1 / \Delta_m^{\alpha+1}$ $\quad p(\Delta) = \dfrac{\alpha+1}{\Delta_m^{\alpha+1}}\Delta^{\alpha}$

for a two level system with gap Δ we have

$$\langle E \rangle = \Delta e^{-\beta\Delta} / 1 + e^{-\beta\Delta} = \Delta / e^{\beta\Delta} + 1$$

$$C = \frac{d\langle E \rangle}{dT} = -k_B\beta^2 \frac{d}{d\beta} \Delta / (e^{\beta\Delta} + 1)$$

$$= +k_B\beta^2 (e^{\beta\Delta} + 1)^{-2} \Delta^2 e^{\beta\Delta}$$

$$= k_B(\beta\Delta)^2 e^{\beta\Delta} / (e^{\beta\Delta} + 1)^2$$

for a collection with prob $p(\Delta)$ we get

$$\overline{C} \langle \epsilon \rangle = \int_0^{\Delta_m} d\Delta\, p(\Delta)\, k_B(\beta\Delta)^2 e^{\beta\Delta} / (e^{\beta\Delta} + 1)^2$$

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$$\langle c \rangle = \int_0^{\Delta_m} \frac{\alpha+1}{\Delta_m^{\alpha+1}} \Delta^\alpha k_B (\beta\Delta)^2 \, e^{\beta\Delta} \Big/ (e^{\beta\Delta}+1)^2 \, d\Delta$$

Defining $x = \beta\Delta$ gives.

$$\langle c \rangle = \frac{(\alpha+1)}{\Delta_m^{\alpha+1}} k_B \int_0^{\beta\Delta_m} \left(\frac{x}{\beta}\right)^\alpha x^2 \frac{e^x}{(e^x+1)^2} \frac{dx}{\beta}$$

$$\langle c \rangle = \frac{\alpha+1}{\Delta_m^{\alpha+1}} k_B \frac{1}{\beta^{1+\alpha}} \int_0^{\beta\Delta_m} x^{2+\alpha} \frac{e^x}{(e^x+1)^2} dx$$

If $T \to 0$ $\beta \to \infty$ and
the integral is T independent

$$\langle c \rangle \sim k_B \left(\frac{k_B T}{\Delta_m}\right)^{1+\alpha}$$

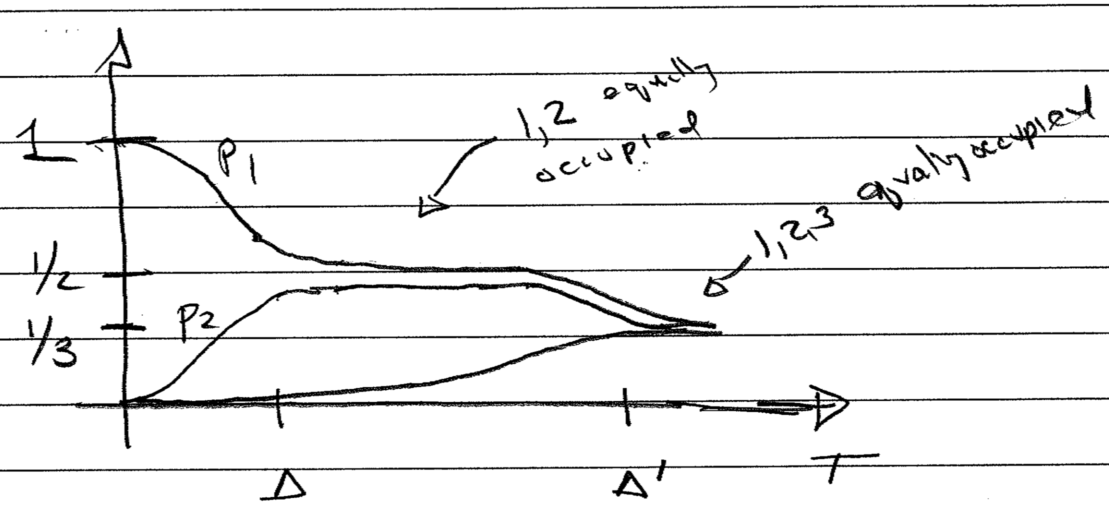
Check: In HW #1 $\alpha = 0$ and $\langle c \rangle \sim T$

In this problem the mystery specific heat with $c \sim T^5$ could be explained if $p(\Delta) \sim \Delta^4$ ie $\alpha = 4$.

3 Ideal relativistic gas $\to$ See solutions to HW #2

4 The crucial observation here is that since $E_3 \gg E_2, E_1$ there are two widely separated energy differences $\Delta = E_2 - E_1$ and $\Delta' \equiv E_3 - E_1 \approx E_3 - E_2$

When $k_B T \approx \Delta$ levels 1 and 2 become equally occupied. When $k_B T \approx \Delta'$ level 3 gets accessed. Thus we expect



An exact result for $\Delta' = 8.8$ $\Delta = 0.4$ is on page 8

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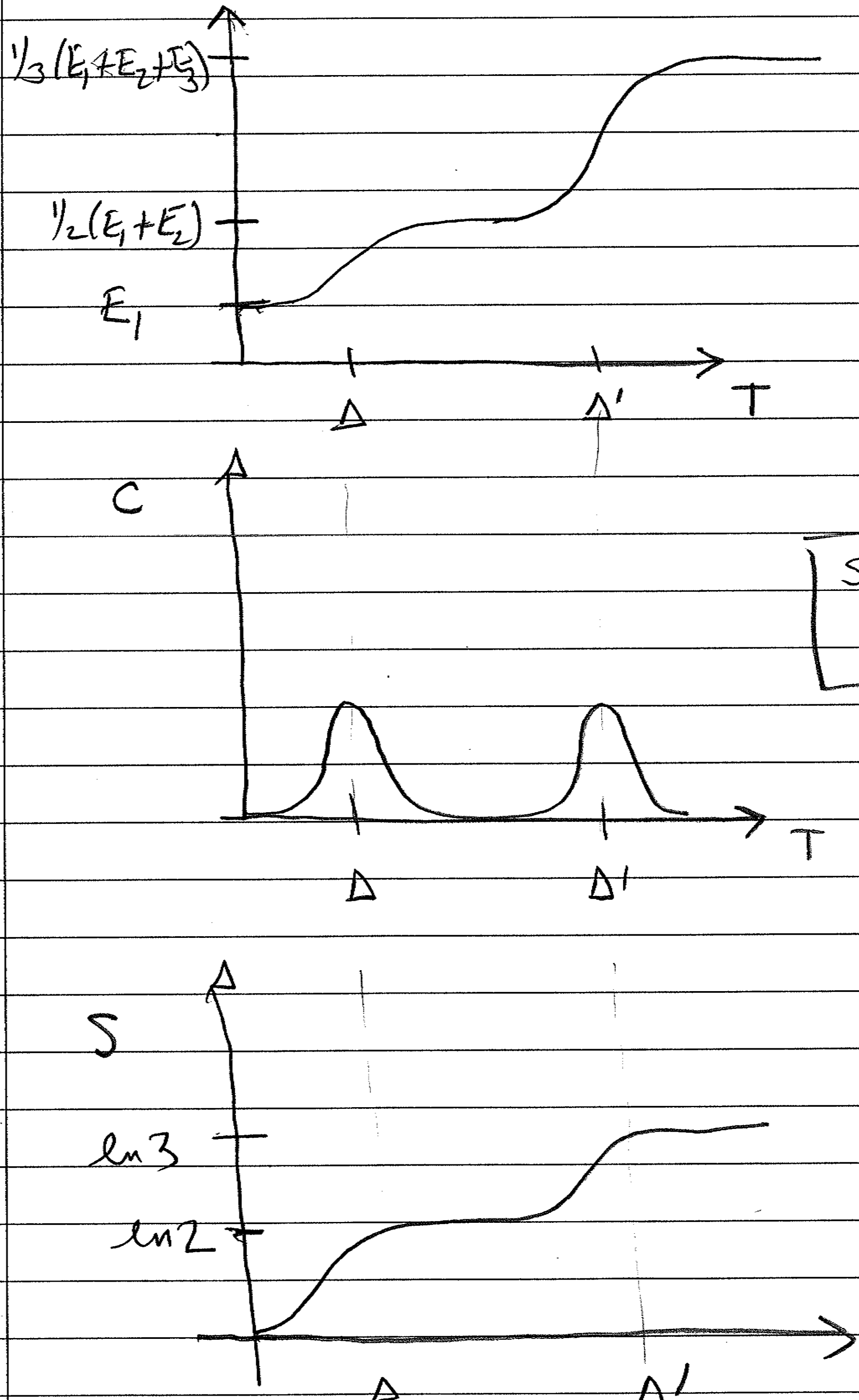
Once p_i are understood it seems clear that



See Also pages 8-11