We found for ideal gas $\hat{H} = \sum \frac{p_i^2}{2m_i}$ that

$$N(E) = V^N \left( \frac{m}{2\pi k}\right)^{3N/2} E^{3N/2 - 1} / \Gamma(3N/2)$$

is total number of states of energy $E$. A key feature of this was rapid growth with $E$.

What do I mean? Suppose you increase energy of system of $10^{23}$ particles by 1 part in a million. How many more states are accessible?

$$\left[ 1 + 10^{-12} \right]^{10^{23}} = \left[ 1 + 10^{-12} \right]^{10^{12} \cdot 10^{11}}$$

$$= e^{10^{11}} \approx a lot!$$

We will come back to this again in a few minutes. First though, $N(E)$ is a less familiar concept. Can we compute something a bit more familiar?

NB: However, we should get familiar with $N(E)$; it is a central quantity in condensed matter physics $\Gamma \sim \langle \text{ler} \rangle^2 N(E)$

$$\left[ 1 + \frac{x}{n} \right]^n = e^x \text{ for } n \text{ large}$$
element: 

\[\begin{array}{c}
\text{Use} \\
\lim_{x \to 0} \frac{\Gamma(\frac{3}{2} - i k^2)}{\Gamma(2)} = e^{x} \\
\text{N}(\vec{p}) = N \frac{4\pi \vec{p}^2}{2\pi m^3} e^{-\frac{\vec{p}^2}{2mE}} \\
= N \left(1 - \frac{\vec{p}^2}{2mE}\right)^{\frac{3}{2} - i k^2} E^{\frac{3}{2} - i k^2} \\
\end{array}\]

\[\begin{array}{c}
\text{Same integral as for N(E)} \\
\text{except N-1 position and energy} \\
= \int d\vec{p} \, d\vec{p}_2 \ldots d\vec{p}_N s_{\vec{E} - \vec{E}_1} s_{\vec{E} - \vec{E}_2} (\vec{p}_3 - \vec{p}_1) \\
\end{array}\]
\[ N(p) = \frac{N \pi (3N/2) \frac{4\pi p^2}{(2\pi m)^{3/2}} \left(1 - \frac{p^2}{2m\xi}\right)^{3/2}(N-1)}{\Gamma(3N/2 - \frac{3}{2}) (2\pi m)^{3/2} \xi^{3/2}} \]

\[ = \left(1 - \frac{p^2}{2m\xi}\right)^{3/2} N (\frac{3N}{2})^{3/2} \frac{(2\pi m)^{3/2} \xi}{(\frac{3}{2})^{3/2} \frac{3N}{2} \xi^{3/2}} \frac{4\pi p^2}{(2\pi m)^{3/2} \xi^{3/2}} \left(1 - \frac{p^2}{2m\xi}\right)^{3/2} \]

\[ \Gamma(3N/2 - 3/2) = \xi \Gamma(\xi) \]

\[ \xi = \frac{\varepsilon}{\sqrt{N}} \]

\[ \int_0^\infty N(p) dp = \frac{N \pi}{(2\pi m)^{3/2}} \xi^{3/2} \int_0^\infty e^{-p^2/2m\xi} dp \]

\[ \int_0^\infty e^{-ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \]

\[ \int_0^\infty x e^{-ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{a^3}} \]

\[ \int_0^\infty x^2 e^{-ax^2} dx = \frac{1}{2} \frac{\sqrt{\pi}}{a^{3/2}} \]

\[ \int_0^\infty e^{-x^2/2m\xi} dx = \frac{\sqrt{\pi}}{\sqrt{2m\xi}} \]

\[ = \sqrt{\frac{\pi}{2}} \frac{\sqrt{2/N}}{2m\xi^{3/2}} \]

\[ = \xi \sqrt{\frac{\pi}{2}} \frac{\sqrt{2/N}}{2m\xi^{3/2}} \]

\[ = \xi \sqrt{\frac{\pi}{2}} \frac{\sqrt{2/N}}{2m\xi^{3/2}} \]

\[ = \xi \sqrt{\frac{\pi}{2}} \frac{\sqrt{2/N}}{2m\xi^{3/2}} \]
\[ N(p) \sim p^2 e^{-\frac{p^2}{2m} \frac{1}{k_B T}} \]

so we would be okay as long as \( \frac{2}{3} \epsilon = k_B T \)

which in fact is okay \( \epsilon = \frac{3}{2} k_B T \).

Let us now consider putting 2 systems into contact so they can exchange energy.

\[ N = N_1 + N_2 \]

\[ N(E) = \int d\epsilon_1 d\epsilon_2 \delta(H_1 + H_2 - E) \]

Probability system 1 has energy \( \epsilon_1 \)

\[ p(\epsilon_1) = \frac{1}{N(E)} \int d\epsilon_2 \delta(H_2 - (E - \epsilon_1)) \]

\[ = \frac{1}{N(E)} \int d\epsilon_1 d\epsilon_2 \delta(H_1 - \epsilon_1) \delta(H_1 + H_2 - E) \]

\[ = \frac{1}{N(E)} N_1(\epsilon_1) N_2(\epsilon_2) \]

\( E_L = E - \epsilon_1 \)

\( \phi(\epsilon_1) \)

\( N_1(\epsilon_1) \)

\( N_2(\epsilon_2) \)

\( \phi(\epsilon_1) \)

\( N_1(\epsilon_1) \)

\( N_2(\epsilon_2) \)
Maximize \( p(E_1) \) or equivalently \( \ln p(E_1) \)

\[
\ln p(E_1) = \ln N_1(E_1) + \ln N_2(E_2) - \ln N(E)
\]

\[
0 = \frac{d}{dE_1} \ln N_1(E_1) - \frac{d}{dE_2} \ln N_2(E_2)
\]

We have discovered a quantity that must be the same for two systems in contact, we will define it to be the inverse temperature

\[
\frac{1}{k_B T} \equiv \frac{d}{dE} \ln N(E)
\]

Reiterating \( T_1 = T_2 \) when systems in contact. Ideal gas

\[
N(E) = c E^{3N/2}
\]

\[
\ln N(E) = \ln c + \frac{3N}{2} \ln E
\]

\[
\frac{1}{k_B T} = \frac{3p}{2E}
\]

Specific heat

\[
E = \frac{3}{2} N k_B T
\]

\[
C = \frac{1}{T} \frac{\partial E}{\partial T} = \frac{3}{2} N k_B
\]

\[
\text{Eqn of state? ??}
\]

So, indeed \( \rho \)

\[
N(p) \sim p e^{-p^2/2 k_B T}
\]
We will talk later at greater length about thermodynamics, but for the moment remember that \( dE = T dS - pdV \).

So \( \frac{1}{T} = \frac{\delta S}{\delta E} \) and \( \frac{1}{k_B T} = \frac{\delta S}{\delta E} \ln N(E) \).

\[ S = k_B \ln N(E) \]

Perhaps this is familiar to you, "entropy is the logarithm of number of states accessible to system."

May as well check out pressure:

\[ \frac{dP}{dV} = \frac{nRT}{V^2} \]
Before abandoning MCE for CE which is much easier to work with, let's do one more example.

\[ N \] degrees of freedom, which can take on values \( \pm 1 \)

\[ H = -B \sum \delta s_i \quad N_\uparrow + N_\downarrow = N \]

\[ E = -B (N_\uparrow - N_\downarrow) = -B (2N_\uparrow - N) \]

\[ N(E) = \frac{N!}{N_\uparrow! N_\downarrow!} \]

\[ \ln N(E) = N \ln N - N_\uparrow \ln N_\uparrow - N_\downarrow \ln N_\downarrow \]

\[ \frac{1}{k_B T} = \frac{2}{E} \left( \frac{\partial}{\partial \ln N_\uparrow} \right) = -\frac{\partial N_\uparrow}{\partial E} \left( 1 + \ln N_\uparrow \right) - \frac{\partial N_\downarrow}{\partial E} \left( 1 + \ln N_\downarrow \right) \]

We also know that

\[ \frac{\partial N_\uparrow}{\partial E} = -\frac{\partial N_\downarrow}{\partial E} \quad \text{since} \quad N_\uparrow + N_\downarrow = N \quad \text{a constant} \]

\[ \frac{\partial N_\uparrow}{\partial E} = -\frac{1}{2B} \]

\[ \frac{1}{k_B T} = \frac{1}{2B} \left( 1 + \ln N_\uparrow \right) - \frac{1}{2B} \left( 1 + \ln N_\downarrow \right) \]

\[ \ln \left( \frac{N_\uparrow}{N_\downarrow} \right) = e^{\frac{2B}{k_B T}} \]

\[ \frac{N_\uparrow}{N_\downarrow} = e^{\frac{2B}{k_B T}} \]
This should again be familiar since we have
two possible choices of Energy $-B$ for $S_z = \uparrow = 1$
and $+B$ for $S_z = \downarrow = -1$ and the difference is
then $dB$ so we learn relative # of # possibilities

$$\sim e^{-\Delta E/k_B T}$$

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**Summarize ME**

1) $P(n) =$ same for all $\Gamma$ of energy $E$

2) Formally $P(n) = \delta(E - H) / \sum \Pi \delta(E - H)$

   Just check if $H(n) = E$ divide by total # of $\Gamma$ satisfying this

3) $N(E) = \sum \Pi \delta(E - H)$

4) $S = k_B \ln N(E)$

Ideal gas example, spin $1/2$ example
Now relate to canonical ensemble

Imagine system 1 is in contact with some much larger system 2. Consider computing the average of some quantity depending on variables in 1.

\[
\langle \hat{O} \rangle = \frac{1}{N(E)} \int d\mathbf{r}_1 d\mathbf{r}_2 \hat{O}_1 \delta(H_1 + H_2 - E)
\]

\[
N(E) = \int d\mathbf{r}_1 d\mathbf{r}_2 \delta(H_1 + H_2 - E)
\]

\[
\langle \hat{O}_1 \rangle = \int d\mathbf{r}_1 \hat{O}_1 \frac{\int d\mathbf{r}_2 \delta(H_1 + H_2 - E)}{\int d\mathbf{r}_1 d\mathbf{r}_2 \delta(H_1 + H_2 - E)}
\]

\[\uparrow\]

\[P_1(\mathbf{r}_1) = \text{factor associated with "integrating out" degrees of freedom of system 2}\]

This factor \(\sim N_2(E - H_1)\)

\[
\ln N_2(E - H_1) = \ln N_2(E) - H_1 \frac{\partial}{\partial E} \ln N_2(E) + \frac{1}{2}H_1^2 \frac{\partial^2}{\partial E^2} \ln N_2(E)
\]

since \(H_1 \ll E\) is small

\[\frac{1}{k_B T} \uparrow \]

\[T_2\]

Compare terms: Ideal gas \(N_2(E) \sim E^{3N/2}\)

\[
\frac{\partial}{\partial E} \ln N_2(E) = \frac{3N}{2E}
\]

\[
\frac{\partial^2}{\partial E^2} \ln N_2(E) \sim -\frac{3N}{2E^2}
\]
\[ \ln N_2(E) = \ln N_2(E - H_1) = C - \frac{H_1}{k_B T} \]

\[ N_2(E - H_1) = c e^{-\frac{H_1}{k_B T}} \Rightarrow p(\pi) \propto e^{-\frac{H_1}{k_B T}} \]

\[ \langle \hat{\theta}_1 \rangle = \frac{\int d\pi_1 \hat{\theta}_1 e^{-\frac{H_1}{k_B T}}}{\int d\pi_1 e^{-\frac{H_1}{k_B T}}} \]

must be this to get normalization correct.

\[ Z(\beta) \quad \beta = \frac{1}{k_B T} \]
Aside: Shannon argument.

"If you know nothing about probability distribution, make all probabilities equal subject to $\sum p_i = 1$"

["Maximum entropy" method in numerical simulations]

"If you know a little make broadest prob. distrib. consistent with that knowledge"

[Again Maxent - know the moments $\phi$ and know $\langle p_c \rangle$ from Monte Carlo at some points]

Shannon's claim is maximize

$$\sum p_i \ln p_i \quad \text{subject to} \quad \sum p_i = 1 \quad \sum p_i A_i = \langle A \rangle$$

accomplishes this

$$f = \sum p_i \ln p_i + \lambda_1 (\sum p_i - 1) + \lambda_2 (\sum p_i A_i - \langle A \rangle)$$

$$\frac{\partial}{\partial p_i} f = \ln p_i + 1 + \lambda_1 + \lambda_2 A_i = 0$$

$$p_i = e^{-\lambda_1 - 1 - \lambda_2 A_i}$$

$$\sum p_i = 1 \quad \Rightarrow \quad p_i = \frac{e^{-\lambda_2 A_i}}{\sum e^{-\lambda_2 A_i}}$$

This is basically what our deriv of CE told us.