Okay, but where did this canonical ensemble come from?! A more natural starting point for such might:

Suppose you are given Hamiltonian $H(\psi, n)$ and energy $E$.

**Microcanonical ensemble:** All states with energy $E$ are equally likely.

\[
P(\psi, n) = \frac{\delta (E - H(\psi, n))}{\int d\psi d\psi n \delta (E - H(\psi, n))}
\]

To normalize count up # of states with energy $E$. Density of states $N(E)$!!
Ideal gas in ME: N free particles in box of volume V

\[ H = \frac{1}{2m} \sum_{i=1}^{N} \frac{p_i^2}{2} = \frac{1}{2m} \sum (p_{x_i} + p_{y_i} + p_{z_i}) \]

\[ N(E) = V^N \int d^3p_1 \int d^3p_2 ... \int d^3p_N \delta(E - \frac{p_1^2}{2m} - ... - \frac{p_N^2}{2m}) \]

2D \int dx dy = 2\pi \int dr \quad \text{In cylindrical, answer depends on length of } Z \text{ dimensioned vector (e.g. analog of spherical coordinates)}

3D \int dx dy dz = 4\pi \int r^2 dr \quad \text{(high d analog of spherical coordinates)}

\[ N(E) = V^N \oint \frac{d^3p}{(2\pi)^3} \delta(E - \frac{p^2}{2m}) \]

\[ \partial \text{ is } 4\pi \quad \text{in } d = 3 \]

\[ \frac{2m \delta(2mE - p^2)}{2m \delta((2mE - p)(2mE + p))} \]

\[ = V^N \Omega_{3N} 2m (2mE)^{(3N-1)/2} \frac{1}{2 \sqrt{2mE}} \]

What the heck is \( \Omega_{3N} \)?

\[ \int d^Dx \ e^{-x^2} = \pi^{D/2} \]

\[ \int_0^\infty x^{D-1} e^{-x^2} \ dx \quad x^2 = t \quad 2x \ dx = dt \]

\[ \frac{1}{2} \int_0^\infty dt \ t^{D/2 - 1} e^{-t} = \frac{1}{2} \Gamma\left(\frac{D}{2}\right) \quad \text{gamma function} \]
\[ \pi_0 = \frac{2\pi^{3/2}}{\Gamma(3/2)} \]

\[ \Gamma(2) = \int_0^\infty dt \, t^{2-1} e^{-t} \]

\[ \Gamma(2+1) = \int_0^\infty dt \, t^{2} e^{-t} \frac{1}{u} \, du \]

\[ = -t^2 e^{-t} \bigg|_0^\infty + \int_0^\infty \left( -t^2 e^{-t} \right) dt \]

\[ \Gamma(2+1) = 2 \Gamma(2) \]

Generalization of factorial function to noninteger \( z \).

Check formula for \( \pi_0 \) in \( D=2, 3 \)

\[ \pi_2 = 2\pi^{3/2} / \Gamma(1) = 2\pi \]

\[ \Gamma(1) = \int_0^\infty dt \, e^{-t} = 1 \]

\[ \pi_3 = 2\pi^{3/2} / \Gamma(3/2) = 2\pi^{3/2} / \pi^{1/2} = 4\pi \vee \]

\[ \Gamma(3/2) = \int_0^\infty dt \, t^{1/2} e^{-t} \quad dt = 2\pi dr \quad t = r^2 \]

\[ = \int_0^\infty 2\pi dr \, r e^{-r^2} = 2\int_0^\infty r^2 dr \, e^{-r^2} \]

\[ = 2 \frac{1}{2} \frac{1}{2} \sqrt{\pi} = \sqrt{\pi} \]
Thus, finally,

\[ N(E) = V^N(2\pi m)^{3N/2}E^{3N/2-1}/\Gamma(3N/2). \]

Much of this nonsense is contained in a Mathematical Appendix to your text.

Let's be math studies for a minute and compute \( N(E) \) another way, the "Laplace transform method". It will turn out that this approach has close connections with the "canonical ensemble" as some of you will recognize from the notation. Define

\[ Z(\beta) = \int d\Gamma e^{-\beta H}. \]

Since \( N(E) = \int d\Gamma \delta(E - H) \), this quantity \( Z(\beta) \) is just the Laplace transform of \( N(E) \)

\[ Z(\beta) = \int_0^\infty d\epsilon e^{-\beta \epsilon} N(E) \]

\[ N(E) = 1/(2\pi i) \int_{a+} -i\infty \) \[ e^{\beta Z(\beta)}e^{\beta E}. \]

But \( Z(\beta) \) is trivial to compute. (This is why working in the canonical ensemble is usually simpler than the microcanonical ensemble).

\[ Z(\beta) = V^N(2\pi m/\beta)^{3N/2} \]

So we compute \( N(E) \) to be (draw a picture of contour of integration, closing the contour off to the left where \( \beta \) has negative real part)

\[ N(E) = 1/(2\pi i) V^N(2\pi m)^{3N/2} \int_{a-} +i\infty \) \[ d\beta \beta^{-3N/2}e^{\beta E} \]

\[ = 1/(2\pi i) V^N(2\pi m)^{3N/2}1/(3N/2 - 1)(d/d\beta)^{3N/2-1}e^{\beta E} \bigg|_{\beta=0} \]

\[ = V^N(2\pi m)^{3N/2}E^{3N/2-1}/\Gamma(3N/2). \]

as before! Whew!

One crucial feature is the rapid growth of \( N(E) \) with \( E \). We will see that this is central to ideas concerning how to systems come into equilibrium with each other.

\[ \text{Problem #4 in HW!} \]
Laplace Transform Review

\[ f(s) = \int_0^\infty e^{-st} F(t) \, dt \]

\[ F(t) = 1 \]

\[ f(s) = \int_0^\infty e^{-st} \, dt = \frac{1}{s} \]

\[ F(t) = e^{wt} \]

\[ f(s) = \int_0^\infty e^{-st} e^{wt} \, dt = \frac{1}{s-w} \]

\[ F(t) = \cos wt = \frac{1}{2} (e^{iwt} + e^{-iwt}) \]

\[ f(s) = \frac{1}{2} \left[ \frac{1}{s-iw} + \frac{1}{s+iw} \right] = \frac{s}{s^2 + w^2} \]

\[ \text{derivatives} \quad \int_0^\infty e^{-st} F'(t) \, dt = e^{-st} F(t) \Big|_0^\infty - \int_0^\infty e^{-st} F(t) \, dt \]

\[ = -F(0) + s \cdot f(s) \]

\[ \text{similiarly} \quad \int_0^\infty e^{-st} F'(t) \, dt = s^2 \cdot f(s) - s \cdot F(0) - F'(0) \]
\[ \text{Solving } \quad mx''(t) = -kx \]

\[ x(t) = x_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t \quad \omega = \frac{k}{m} \]

\[ s^2 X(s) - sx(0) - v(0) = -\frac{k}{m} X(s) \]

\[ \frac{X(s)}{s^2 + \omega^2} = sX(0) + v(0) \]

\[ X(s) = X(0) \frac{s}{s^2 + \omega^2} + \frac{v(0)}{\omega^2} \frac{\omega}{s^2 + \omega^2} \]

\[ x(t) = x(0) \cos \omega t + \frac{v(0)}{\omega} \sin \omega t. \]

\[ \text{Inverting } \quad F(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} f(s) ds \]

\[ \gamma \text{ is a value such that all singularities of } f(s) \]

\[ \text{are to left} \]

\[ f(s) = \frac{1}{s} \]

\[ F(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} \frac{1}{s} ds = \frac{e^{st}}{s} \]

\[ \text{ pole at } s = 0 \]

\[ \frac{1}{2\pi i} \text{ Residue } = 1 \]

\[ F(t) = 1 \]
\[ f(s) = \frac{1}{s - \omega} \quad \text{pole at} \quad s = \omega \quad \omega > 0 \]

\[
\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} \frac{1}{s - \omega} \, ds
\]

\[
e^{\omega t} (s - \omega)^{-1} \frac{1}{s - \omega} \left[ 1 + (s - \omega)t + \frac{1}{2} (s - \omega)^2 t^2 + \cdots \right]^{-1}
\]

\[ \text{Residue} = e^{\omega t} \]

\[ f(t) = e^{\omega t} \quad \forall \]