We had \( \langle N \rangle = \frac{3}{2} \frac{V}{(\lambda_T)^3} \)

\[
\lambda_T = \frac{\hbar}{(2\pi \hbar m k_B T)^{1/2}}
\]

\[
= 6 \times 10^{-34} / \left( \frac{2 \pi \times 1.67 \times 10^{-27} \times 1.38 \times 10^{-23} \times 300}{1} \right)^{1/2}
\]

\[
= 1.6 \times 10^{-11} \text{ m}
\]

\( l_{\text{m, solid}} \sim 1 \AA = 10^{-10} \text{ m} \)

\( l \text{ in an at STP} \sim 10^8 \AA \sim 10^{-9} \text{ m} \)

\[
\lambda_T / l \sim 10^{-2} \quad \Rightarrow \mu \text{ is large and negative}
\]

\[
e^{-\beta \mu} \sim 10^{-2}
\]

\( E_i = \frac{p^2}{2m} > 0 \)

\( \mu \sim -4.6 \text{ k_B T} \)

Recall \( \langle E \rangle = \frac{3}{2} k_B T \)

\[
\Rightarrow e^{-\beta (\mu - E_i)} < 10^{-2}
\]
\[ \text{Solid} \quad \rho \approx 1 \text{ g/cm}^3 \]

\[ \approx 1 \text{ g/cm}^3 \quad \frac{6 \times 10^{23} \text{ atoms}}{18 \text{ g}} = \frac{10^{23} \text{ atoms}}{3 \text{ cm}^3} \]

\[ \frac{1}{8^3} \approx \frac{10^{24}}{30} \quad \frac{1}{8} \approx 10^{8/3} \text{ cm} \]

\[ \frac{1}{2} \approx 10^{10/3} \text{ m} \]

\[ l \approx 3 \cdot 10^{-10} \text{ m} \text{ for } H_2O \text{ liquid} \]

\[ \text{Gas} \quad \frac{PV}{N} = k_B T \]

\[ 10^5 \left( \frac{1.3 \cdot 10^{-23}}{1.3 \cdot 10^5} \right) = \frac{N}{\sqrt{V}} \]

\[ 1 \text{ atom} = 10^5 \frac{N}{\text{m}^2} \]

\[ \frac{10^{26}}{4} \approx \frac{N}{V} \quad \frac{10^{27}}{40} \quad 8 \approx \frac{10^9}{3} \]

\[ l \approx 3 \cdot 10^{-7} \text{ m} \]
Discrete Energy levels $E_1, E_2, \ldots$

Classically

\[ N \quad Z_N \quad e^{\beta \mu N} \quad \frac{1}{N!} \]

0 \quad 1 \quad 1 \quad 1

1 \quad \left( e^{-\beta E_1} + e^{-\beta E_2} + \ldots \right) \quad e^{\beta \mu} \quad 1

2 \quad \left( \frac{1}{2} \right)^2 e^{2\beta \mu} \quad \frac{1}{2}!

\[ Q = \sum_{N=0}^{\infty} Z_N (\sum_i e^{-\beta E_i})^N / N! = e^{3z_1} \]

with fugacity $Z = e^{\beta \mu}$

\[ z_1 = \sum_i e^{-\beta E_i} \]

\[ \mathcal{Z} = -\frac{1}{\beta} \ln Q = -\frac{1}{\beta} \ln z_1 = -\frac{1}{\beta} \sum_i e^{\beta (\mu - E_i)} \]

\[ \langle N \rangle = \frac{1}{\beta} \frac{\partial}{\partial \mu} \ln Q = -\frac{\partial \mathcal{Z}}{\partial \mu} \]

\[ \frac{\partial}{\partial \mu} \sum_i e^{\beta (\mu - E_i)} = \sum_i \frac{d}{d \mu} e^{\beta (\mu - E_i)} = \sum_i \eta_i \]

Small for classical system
\[ N \rightarrow Z \rightarrow \text{No!} \]

\[ \begin{array}{c}
0 & 1 & 1 \\
1 & e^{-\beta E_1} & e^{-\beta E_2} + \cdots & e^{-\beta \mu} \\
2 & e^{-\beta (E_1 + E_2)} & e^{-\beta (E_1 + E_2)} & e^{-2\beta \mu} \\
& & + \cdots & \\
\end{array} \]

\[ Q = 1 + e^{\beta (M-E_1)} + e^{\beta (M-E_2)} + e^{\beta [M-E_1 + M-E_2]} + \cdots \]

\[ = \prod_{i=1}^{N} \left( 1 + e^{\beta (M-E_i)} \right) \]

\[ \text{Clearly each } E_i \text{ can appear only 1 time in product.} \]

\[ \Omega = -\frac{1}{\beta} \ln Q = -k_B T \sum_i \ln \left( 1 + e^{\beta (M-E_i)} \right) \]

\[ \langle N \rangle = \frac{\partial \Omega}{\partial \mu} = \sum_i \left( 1 + e^{\beta (M-E_i)} \right) - 1 \cdot e^\beta (M-E_i) \]

\[ \text{Max.} = \sum_i \frac{1}{e^{\beta (E_i - \mu)} + 1} \]

\[ \text{Indep. sum over } \mu_i \text{ Fermi-Dirac Distribution!} \]

\[ \text{GCE fermions} \]
GCE 9

GCE Bosons

\[ N \sim e^{\beta \mu N} \]

\[ 0 \equiv 1 \]

\[ 1 \equiv e^{-\beta E_1} + e^{-\beta E_2} + \cdots \equiv e^{-\beta \mu} \]

\[ 2 \equiv e^{-2\beta E_1} + e^{-\beta (E_1 + E_2)} e^{2 \beta \mu} \]

\[ + e^{-2\beta E_2} + \cdots \]

\[ Q = (1 + e^{\beta (\mu - E_1)} + e^{2 \beta (\mu - E_1)} + \cdots) \]

\[ (1 + e^{\beta (\mu - E_2)} + e^{2 \beta (\mu - E_2)} + \cdots), \ldots \]

\[ = \prod_i \left( 1 - e^{\beta (\mu - E_i)} \right) \]

\[ \uparrow \quad \text{Signs differ from Fermions} \]

\[ \Lambda = -\frac{1}{k_B T} \ln Q = \frac{1}{k_B T} \sum_i \ln \left( 1 - e^{\beta (\mu - E_i)} \right) \]

\[ \uparrow \quad \text{Signs differ from Fermions} \]

\[ \langle N \rangle = \frac{-\Lambda}{\delta \mu} = \sum_i \frac{e^{\beta (\mu - E_i)}}{1 - e^{\beta (\mu - E_i)}} = \sum_i \frac{1}{e^{\beta \mu - (\mu - E_i)} - 1} \]

\[ \text{Bose-Einstein distribution} \]
Anyons, Spin-charge Separation

Topological excitations

**FD**: \[ Q = \frac{\pi}{i} \left( 1 + e^{\beta (n + e i)} \right) \]

**BE**: \[ Q = \pi \left( 1 - e^{\beta (n - e i)} \right)^{-1} \]

Natural generalization

\[ Q = \frac{\pi}{i} \left( 1 + e^{i\phi} e^{\beta (n - e i)} \right)^{-1} \]

Fermions \( \phi = 0 \), \( e^{i\phi} = 1 \)

Bohons \( \phi = \pi \), \( e^{i\phi} = -1 \)

Anyons, general \( \phi \)

Actually such "complex generalizations" occur in various contexts in stat mech

\[ H = -J \sum s_i s_j - B \sum s_i \]

\[ Z = \sum e^{-\beta H(\{s\})} \]

Ising model: Allow

\( B \) to be a complex #

and look for poles of \( Z \)

\( \text{Im}(B) \)

in complex plane, eg on

finite lattice size \( N \)

"Lee Yang Theorem": Poles of \( Z \) only for

"Lee-Yang Zeros" \( \text{Imaginary } B \)

As \( N \) increases poles pinch
down towards \( B = 0 \) signaling

the phase transition
Tied to some of most interesting issues
in modern CMP. Eg "spin-change separation."

How a single added particle can split into two
separate excitations: Spinless fermions with no repulsion

ADD PARTICLE

Allow to move
More interesting if spin involved

Antiferromagnet

\[
\uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow
\]

Photon knocks one e\textsuperscript{-} out

\[
\uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow
\]

Again, allow time to pass create hole spin \( \frac{1}{2} \) of spin \( \uparrow \)

\[
\uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow
\]

"spin\( \uparrow \)" "hole\( \uparrow \)"

Spin \( \frac{1}{2} \) Spin \( \uparrow \)

Charge 0 Charge 1
We emphasized GCE as easier computationally especially for quantum systems. Another motivation is that it is the correct way to do things if there really are fluctuations, e.g. with 2 systems in contact.

Just as $T_1 = T_2$ thermal equilibrium

$N_1 = N_2$ "chemical equilibrium"

Indeed exact same proof as before for $T_1 = T_2$. You have HW problem on this.

Example: Classical 3D ideal gas at pressure $p, T$

above a surface in which there are bound states of energy $-E_1$. Q: How many surface states are occupied?

Each area $A_0$ contains bound state $N_0 = A/a_0 = \# \text{ bound states}$
Set mech of surface states.

Two ways give same answer.

Method #1

\[ N \quad 3^N \quad E \quad \frac{1}{N!} \]

\[ N \quad 3 \quad -E_1 \leq N \text{ of these } \frac{1}{2}! \]

\[ 2 \quad 3^2 \quad -2E_1 \quad N_0(N_0-1) \text{ of these } \frac{1}{2}! \]

\[
Q = \sum_{N=0}^{N_0} 3^N e^{\beta E_1/N_0} \cdot \frac{1}{N_0} \cdot \frac{1}{(N_0-N)!} \cdot \frac{1}{N!}
\]

\[ Z_N \text{ because all states have same } E = -N E_1 \]

and there are \( N_0! / (N_0-N)! \) of them

\[
= \sum_{N=0}^{N_0} e^{\beta(\mu+E_1)N} \left( \frac{N_0}{N} \right) \left( \frac{N}{N_0} \right)
\]

\[
= \left( 1 + e^{\beta(\mu+E_1) N_0} \right)^{-N_0}
\]
Alternatively have N₀ systems each with GCE

\[ 1 + e^\beta (\mu + E_1) \]

so

\[ Q_S = \left( 1 + e^\beta (\mu + E_1) \right)^{N_0} \]

\[ \Delta S = -k_B T \ln Q_S = -N_0 k_B T \ln \frac{1}{1 + e^\beta (\mu + E_1)} \]

For ideal gas above the surface

\[ Q = \frac{V}{V^*} \]

\[ \Delta V = -k_B T \frac{V}{V^*} \frac{V}{V^*} \]

\[ p = -\frac{\Delta P V}{\Delta V} = \frac{k_B T}{V^*} \]

\[ p V = k_B T \frac{V}{V^*} \]

Surface

\[ \left< N \right> = -\frac{\Delta S}{\Delta \mu} = N_0 \frac{1}{1 + e^\beta (\mu + E_1)} \]

\[ = N_0 \frac{1}{e^{-\beta (\mu + E_1)} + 1} \]

\[ e^{-\beta \mu} = \frac{1}{3} = \frac{k_B T}{p \lambda_T^3} \]

\[ \left< N \right> = N_0 \frac{k_B T}{p \lambda_T^3} e^{-\beta E_1} + 1 \]
Think about structure of this result.

What do you expect for $E_i \to \infty$ (very high bond?)

$\langle N_5 \rangle \to N_0$ as expected

What about high pressure $P$?

$\langle N_5 \rangle \to N_0$ as expected at high $P$

$\to 0$ at low $P$ (as expected)

Temperature dependence is less obvious?

$T \to \infty$ $\beta \to 0$ $e^{-\beta E_i} \to 1$

$$\frac{k_b T}{(\hbar/2\pi m k_B T)^{1/2}} \sim T^{5/2}$$

High $T$ $\langle N \rangle \to 0$ "desorption"

Two effects $E_i$ becomes less effective at binding atoms

$e^{-\beta E_i} \to 0$ but also more "phase space" for 3D gas above surface.
Application of GCE formalism to fermions

Fermi gas at temperature $T$  

$$E_k = \frac{k^2}{2m}$$

$$N = 2 \sum_k \frac{1}{e^{E_k - \mu} + 1}$$

for spin

$$N = 2 \frac{V}{(2\pi)^3} \int d^3k \frac{1}{e^{E_k - \mu} + 1}$$

If $\beta \rightarrow 00$ ($T \rightarrow 0$)

$$N = 2 \frac{V}{(2\pi)^3} \frac{4}{3} \pi k_F^3$$  

where $k_F$ is largest $k$ which is occupied

$$\mu = E_F = \frac{k_F^2}{2m}$$

$$n = \frac{N}{V} = \frac{k_F^3}{\frac{4}{3} \pi \alpha^2}$$

$e^-$ density.
Take a typical metal like sodium

\[ n = \frac{N}{V} = \frac{6 \times 10^{23}}{23 \times 10^{-6}} \text{ m}^3 \]
Density of sodium is \( \rho = 1 \text{ g/cm}^3 \)

So \( 23 \text{ g of sodium (1 mole) occupy } 23 \text{ cm}^3 \)

\[ k_F^3 = \frac{\pi^2}{2} \frac{6}{23 \times 10^{-6}} \approx 0.77 \times 10^{30} \]

\[ k_F = 0.92 \times 10^{-10} \text{ m} \]

\[ E_F = \frac{k_F^2}{2m} = \frac{(1 \times 10^{-34})^2 (10^{10})^2}{2 (9 \times 10^{-31})} = \frac{1}{2} = 10^{-68} + 20 + 30 = \frac{1}{2} \times 10^{18} \]

\[ 1 \text{ eV} = 1.6 \times 10^{-19} \text{ J} \]

\[ E_F = 3 \text{ eV} \approx 36600 \text{ K} \]

\[ \frac{E_F}{k_B} = \frac{\frac{1}{2} \times 10^{-18}}{1.38 \times 10^{-23}} \approx 1.16 \times 10^5 \text{ K} \]

Anyway, the point is electrons have a huge kinetic energy. This is why treating metals as a free electron gas (ignoring interactions) is a good starting point.
In a CMP course you will learn how to handle finite $T$.

The method is called the "Sommerfeld Expansion" and is an expansion in $k_B T / E_F$ which we just saw is a very small term.

Basic identity (see CMP text for derivation)

$$
\int_{-\infty}^{\infty} H(E)f(E) \, dE = \sum_{-\infty}^{\infty} H(E) \, dE + \frac{\pi^2}{6} (k_B T)^2 \, H'(E_0) + \ldots
$$

Fermi function

Can use this to show $C \propto T$ for a gas of electrons.

Imp: also to astrophysics: very high KE of neutrons in collapsed neutron star prevents further gravitational collapse.

"degeneracy pressure"
Bose-Einstein Condensation

\[ N = \sum_p n_{p \epsilon (p)} \]
\[ E_p = \frac{p^2}{2m} \]
\[ = \sum_p \frac{1}{e^{E_p - \mu} - 1} \]
\[ = \frac{V}{(2\pi \hbar)^3} \int \frac{d^3p}{p^3} \frac{1}{e^{E_p - \mu} - 1} \]

In order to increase \( N \) we move \( \mu \) up towards the lowest energy state \( E = 0 \).

\[ E(p) \uparrow \]
\[ N = \frac{4\pi}{V} \int_0^\infty \frac{p^2 dp}{e^{p^2/(2m) - \mu} - 1} \]