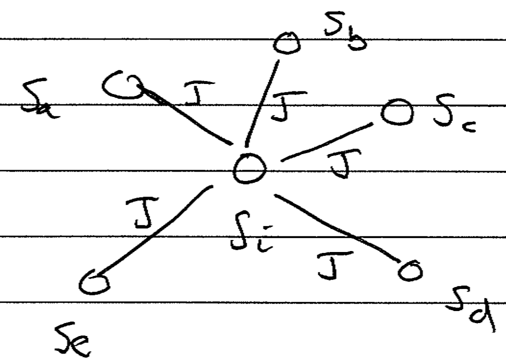
MFT I

MEAN FIELD THEORY

Crude derivation



Spin S_i interacts with

z neighbours $\{S_a, S_b, S_c \dots\}$

↑

"coordination #"

Replace $S_a, S_b, S_c \dots$

by their average value

$m = \langle S_a \rangle$

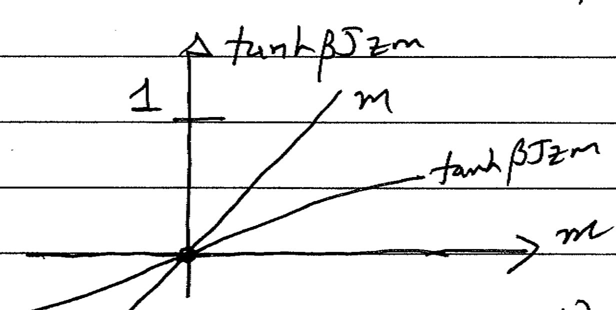
$$-J\, S_i (S_a + S_b + S_c + \dots) \longrightarrow -J z m S_i$$

Then S_i is noninteracting. Can compute

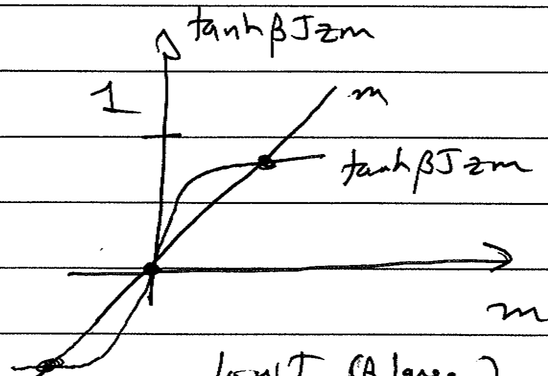
$$m = \langle S_i \rangle = \tanh \beta J z m$$

Self consistent eqn for m.

Has sol'n $m = 0$ obviously!



high T (β small)



low T (β large)

MFT2

Only sol'n to $m = \tanh \beta J m z$ is $m=0$ at high T.

At low T there are 3 solns. Critical temperature

is when $\tanh \beta J z m$ has slope ≥ 1 at $m=0$

$$\beta J z \geq 1$$

$$k_B T \leq J z \qquad T_c = Jz/k_B \text{ in MFT.}$$

Notice MFT incorrectly predicts transition even in $d=1$!!

Ising Model

dim	MFT T_c	actual T_c	$\frac{\text{actual}}{\text{MFT}}$
1	$2J$	0	.000
2	$4J$	$2.269J$	.567
3	$6J$	$4.51J$	.752

MFT becomes exact in $d=\infty$

plausible: when S_i interacts with many (z) neighbors it is more valid to do replacement by average

MFT3

To see $m \neq 0$ is lower free energy,

do more careful analysis

$$S_i \to m + (S_i - m)$$

$$E = -J \sum_{\langle ij \rangle} S_i S_j \to -J \sum_{\langle ij \rangle} [m + (S_i - m)][m + (S_j - m)]$$

$$= -J \sum_{\langle ij \rangle} m^2 + m(S_j - m) + m(S_i - m) + \underbrace{\cancel{(S_i - m)(S_j - m)}}_{\text{neglect}}$$

$$= +J \sum_{\langle ij \rangle} m^2 - 2Jm \sum_{\langle ij \rangle} S_i$$

$$= +\underbrace{Jz/2\, N}_{\substack{\text{\# of bonds } \langle ij \rangle \\ \text{is } Nz/2}} m^2 - \underbrace{Jzm \sum_i S_i}_{\text{previous expression}}$$

sites → N, coordination # → z

MFT 4

Free energy $F = -k_B T \ln Z$ ← partition function (not coordination #)

$$Z = \sum_{S_1 S_2 \ldots S_N} e^{-\beta E} = \sum_{S_1 S_2 \ldots S_N} e^{-\beta J z/2\, N m^2} e^{\beta J z m \sum S_i}$$

$$= e^{-\beta J z/2\, N m^2} \left(2\cosh \beta J z m\right)^N$$

$$f = \frac{E}{N} = \frac{Jz}{2} m^2 - k_B T \ln\left(2\cosh \beta J z m\right)$$

For T near T_c m will be small

$$\cosh x \simeq 1 + x^2/2 + x^4/24$$

$$\ln(\cosh x) = \ln\left(1 + \frac{x^2}{2} + \frac{x^4}{24} \ldots\right) = \frac{x^2}{2} + \frac{x^4}{24} - \left(\frac{x^2}{2} + \frac{x^4}{24} \ldots\right)^2 = x^2/2 - 5x^4/24$$

$$f = \frac{Jz}{2} m^2 - k_B T \ln 2 - k_B T\left(\frac{(\beta J z m)^2}{2} - \frac{5(\beta J z m)^4}{24} \ldots\right)$$

$$f = -k_B T \ln 2 + \frac{Jz}{2} m^2 \left(1 - \beta J z\right) + \frac{5Jz}{24} m^4 (\beta J z)^3$$

high T $m=0$ and $\langle E \rangle = 0$

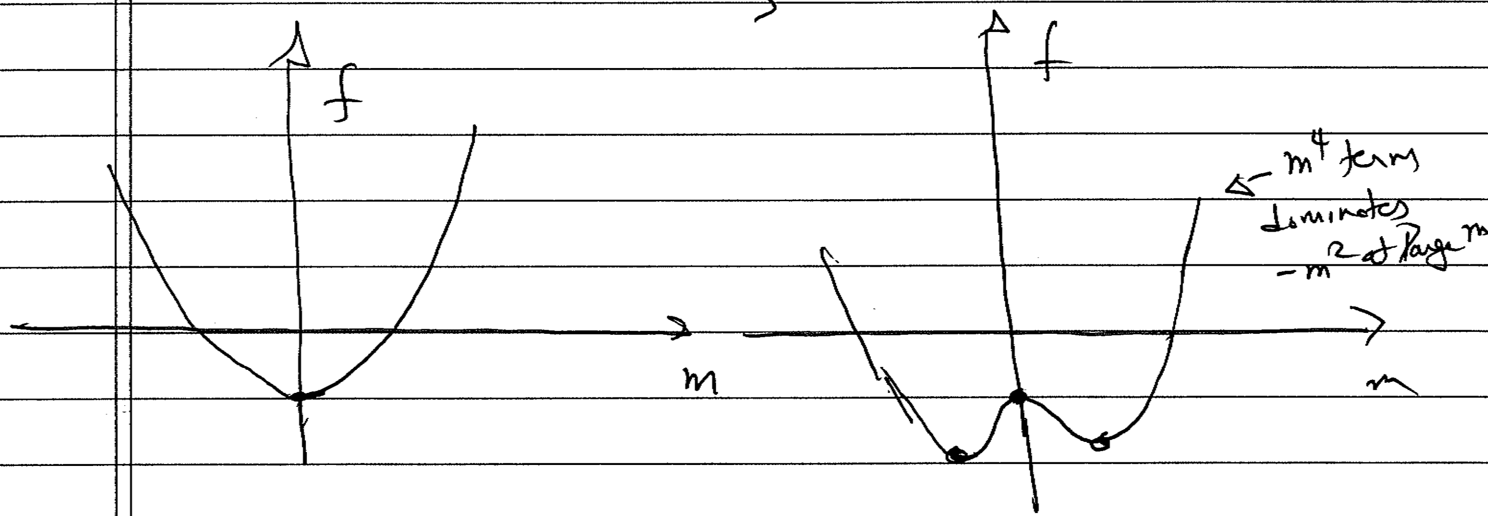
$f = \langle E \rangle - TS \to -T(k_B \ln 2)$ ✓✓

$$\left(1 - Jz/k_B T\right)$$

MFT 5

$k_B T > Jz$ $\quad m^2$ term has + coeff

$k_B T < Jz$ $\quad m^2$ term has − coeff



Clearly $T_c = Jz/k_B$ as before!

Now we see $m \neq 0$ is lowest free energy

"Landau MFT"

We did this calculation explicitly for Ising but Landau said that for <u>any</u> phase transition one can expand free energy as a power series in some "order parameter" m.

$$f(m) = f_0 + a m^2 + b m^4$$

a and b will be functions of temperature and phase transition occurs when $a(T)$ changes sign and becomes negative. We saw this is precisely the situation for Ising. For T near T_c:

$$f(m) = f_0 + a_0 (T - T_c) m^2 + b(T_c) m^4$$

{Since $a(T_c) = 0$ by assumption, replace $a(T)$ by first nonzero term in Taylor expansion. Assume $b(T)$ is "uninteresting" and replace by $b(T_c)$}

MFT7

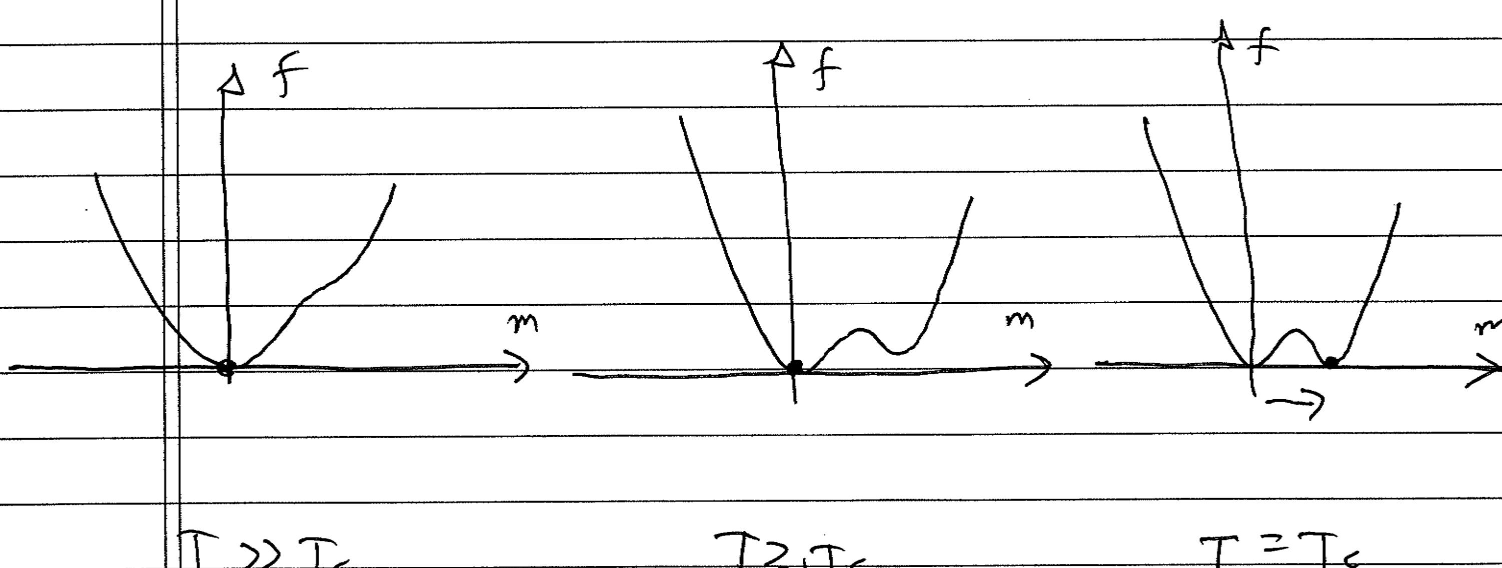
$$\frac{\partial f}{\partial m} = 2a_0(T-T_c)m + 4m^3 b(T_c)$$

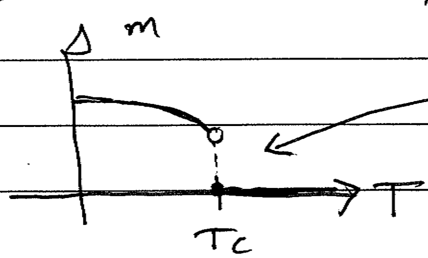
$$m = 0 \quad \text{or} \quad m = \left[\frac{a_0(T_c-T)}{b(T_c)}\right]^{1/2}$$

We get our first critical exponent

$$m \sim (T_c - T)^\beta \qquad \beta = 1/2 \text{ in MFT}$$

Landau even allows understanding of 1st order phase transitions! Suppose there is a cubic term also in $f(m)$ (Ising doesn't have this)





Sudden jump in m at $T = T_c$