I 15

Low T expansions

$$Z = \sum_{S_1 S_2 \ldots S_N} e^{-\beta E}$$

lowest E is
$$\begin{matrix} + & + & + & + & + \\ + & + & + & + & + \\ + & + & + & + & + \\ + & + & + & + & + \end{matrix} = -2NJ$$

$$Z = 2e^{2\beta NJ}$$

Next lowest: Flip a spin

on four bonds $-J$ goes to $+J$

$$\begin{matrix} + & + & + & + & + \\ + & + & + & - & + \\ + & + & + & + & + \\ + & + & + & + & + \end{matrix}$$

So E is $8J$ higher

Flipped spin could be <u>anywhere</u>

$$Z = 2e^{2\beta NJ} + 2e^{2\beta NJ} N e^{-8J\beta}$$

Then flip two adjacent spins.

"Break" six bonds

$$\begin{matrix} + & + & + & + \\ + & - & - & + \\ + & + & + & + \end{matrix}$$

$$Z = 2e^{2\beta NJ}\left\{1 + Ne^{-8J\beta} + 2Ne^{-12\beta J} + \ldots\right.$$

look familiar?

Flip 3 spins

$$\begin{array}{ccccc} + & + & + & + & + \\ + & - & - & - & + \\ + & + & + & + & + \end{array}$$

or

$$\begin{array}{cccc} + & + & + & + \\ + & - & - & + \\ + & + & - & + \\ + & + & + & + \end{array}$$

Both break 8 bonds

$\Delta E = 16J$

look familiar?

$Z_{low\,T}$ has same form as $Z_{high\,T}$ with

expansion parameter $\tanh \beta J \leftrightarrow e^{-2\beta J}$

Proof of identical nature of high T and low T:

Do construction: If bond is "broken" (adjacent spins antiparallel) draw perpendicular bisector.

$$\begin{array}{ccc} + & + & + \\ + & \boxed{-} & + \\ + & + & + \end{array} \qquad \begin{array}{cccc} + & + & + & + \\ + & - & - & + \\ + & + & - & + \\ + & + & + & + \end{array}$$

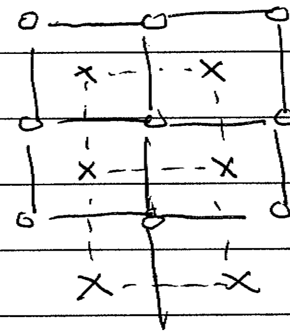
get high T diagrams from low T ones

Square lattice is "self-dual"

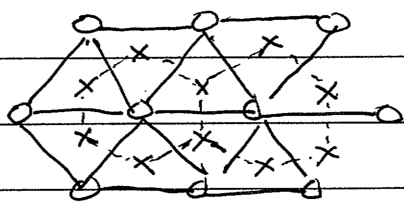
Dual lattice:

o original

x dual



square ↔ square



triangular ↔ hexagonal

Related to basis vectors in real space (giving atomic positions) and momentum space (giving Bragg scattering vectors)

Self-dual lattices can yield interesting further information about physics eg phase transitions.

I-18

Z is nonanalytic $\longleftrightarrow$ phase transition

$1 + Nt^4 + 2Nt^6 + \ldots$ must "go bad" at some value of t

$1 + Ne^{-8\beta J} + 2Ne^{-12\beta J} + \ldots$ " " " " $e^{-2\beta J}$

Assuming just one "bad" value

$$\tanh \beta J = e^{-2\beta J}$$

$$\frac{e^{\beta J} - e^{-\beta J}}{e^{\beta J} + e^{-\beta J}} = e^{-2\beta J}$$

$$\frac{e^{2\beta J} - 1}{e^{2\beta J} + 1} = e^{-2\beta J} \qquad x \equiv e^{2\beta J}$$

$$\frac{x-1}{x+1} = \frac{1}{x}$$

$$x^2 - x = x + 1$$

$$x^2 - 2x - 1 = 0$$

$$x = \frac{1}{2}\left[2 \pm \sqrt{4+4}\right]$$

$$= 1 + \sqrt{2}$$

$$e^{2\beta J} = 1 + \sqrt{2}$$

$$2\beta J = \ln(1+\sqrt{2})$$

$$\frac{k_B T_c}{J} = \frac{2}{\ln(1+\sqrt{2})}$$

$$= 2.269185$$

!!

We have T_c for 2D Ising model !!