Low T expansions

\[ Z = \sum_{\delta_1, \delta_2, \ldots, \delta_N} e^{-\beta E} \]

\[ \text{lowest } E \text{ is } \quad \quad \quad = -2N J \]
\[ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + + + + + \]
\[ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + + + + + \]

\[ Z = 2 e^{2 \beta N J} \]

Next lowest: Flip a spin \[ + + + + + \]

on four bonds \[ -J \text{ goes to } +J \quad + + + + + \]

so \[ E \text{ is } 8J \text{ higher} \]

Flipped spin could be anywhere \[ \frac{1}{2} \]

\[ Z = 2 e^{2 \beta N J} + 2 e^{2 \beta N J} N e^{-8J \beta} \]

Then flip two adjacent spins.

Break six bonds \[ + + + + + \]
\[ + \quad \frac{-J}{-J} \]
\[ + + + + + \]
\[ Z = 2e^{2\beta N J} \left[ 1 + Ne^{-\beta J} + 2Ne^{-2\beta J} + \cdots \right] \]

Look familiar?

Flip 3 spins

\[ + + + + \]
\[ + - - + \]
\[ + + + - \]

or

\[ + + + + \]
\[ + - - + \]
\[ + + - + \]
\[ + + + + \]

Both break & bond

\[ \Delta E = 16J \]

Look familiar?

\[ Z \text{ but has same form as } Z_{\text{high T}} \text{ with} \]

Expansion parameter \[ \tanh \beta J = \exp(-2\beta J) \]

Proof of identical nature of high T and low T:

Do construction: If bond is "broken" (adjacent spins antiparallel) draw perpendiculars bisector.

\[ + + + + \]
\[ + + + + \]
\[ + + + + \]
\[ + + + + \] get high T diagrams
\[ + + + + \]
\[ + + + + \] from low T ones
Square lattice is "self-dual"

Dual lattice:

Original:

Dual:

Square → Square

Triangular → Hexagonal

Related to basis vectors in real space (giving atomic positions) and momentum space (giving Bragg scattering vectors)

Self-dual lattices can yield interesting further information about physics and phase transitions.
$Z$ is nonanalytic $\iff$ phase transition

$1 + N t^4 + 2 N t^6 + \ldots$ must "go bad" at some value of $T$

$1 + N e^{-\beta J} + 2 N e^{-2\beta J} + \ldots = e^{-2\beta J}$

Assuming just one "bad" value

$\tanh \beta J = e^{-2\beta J}$

$\frac{e^{\beta J} - e^{-\beta J}}{e^{\beta J} + e^{-\beta J}} = e^{-2\beta J}$

$e^{2\beta J} - 1 = e^{-2\beta J} = e^{-2\beta J}$

$X = e^{2\beta J}$

$\frac{X - 1}{X + 1} = \frac{1}{X}$

$2\beta J = 2 \ln(1 + \sqrt{2})$

$x^2 - x = x + 1$

$x^2 - 2x - 1 = 0$

$x = \frac{1}{2} \left[ 2 \pm \sqrt{4 + 4} \right]

x = 1 + \sqrt{2}$

$K_B T_c = \frac{2}{J} \ln(2(1 + \sqrt{2})) = 2.269185$

We have $T_c$ for 2D Ising model.