High temperature expansion method

\[ e^{\beta S_i S_j} = \cosh\beta J + S_i S_j \sinh\beta J \]

for \( S_i S_j = \pm 1 \)

\[ = \cosh\beta J \left[ 1 + S_i S_j \tanh\beta J \right] \]

\[ Z = \sum_{S_1 S_2 \ldots S_N} < \prod_{i \neq j} e^{\beta J S_i S_j} \]

\[ = \sum_{S_1 S_2 \ldots S_N} < \prod_{i \neq j} \cosh\beta J \left( 1 + S_i S_j \tanh\beta J \right) \]

\[ = (\cosh\beta J)^N \sum_{S_1 S_2 \ldots S_N} < \prod \left( 1 + S_i S_j \tanh\beta J \right) > \]

\[ N \text{ terms} \]

\[ \ln \frac{1}{\langle \cosh \beta J \rangle} \]
Next geometrical construction:

on each bond you have your choice of

\( 1 \) or \( \frac{s_i s_j \tanh \beta J}{s_i s_j} \) from \( \prod (1 + \frac{s_i s_j \tanh \beta J}{s_i s_j}) \)

at high \( T \), \( \beta \) is small and so \( \tanh \beta J \) so

best to take \( 1 \) all the time.

First term in high \( T \) expansion

\[
Z = (\cosh \beta J)^N \sum_{s_i s_2 \ldots s_N} \prod_{i=1}^{N-1} 1
\]

\[
\Rightarrow (2 \cosh \beta J)^N
\]

Next biggest term: choose 1 from all bonds but one, suppose \( s_5 s_6 \)

\[
Z = (\cosh \beta J)^{N-1} \sum_{s_i s_2 \ldots s_N} s_5 s_6 \tanh \beta J
\]

\[
\text{vanishes since } \sum_{s_5} s_5 = \phi
\]

\[
\sum_{s_5}
\]
\[ \sum s_5 = 0 \]
\[ \sum s_5^2 = 2 \]
\[ \sum s_y \left( \cosh \beta J \right)^N \]
\[ \sum s_5 s^5_5 s_5 s_5 \left( \tanh \beta J \right)^2 \]
\[ s_5 s_5 \cdots s_N \]

But \( s_4 s_6 \) continue to give \( \phi \).

Must include all the \( s_i s_j \tanh \beta J \) and \( n = 1 \)
to get nonzero result. Only other contribution to \( Z \)
is \[ \left( \cosh \beta J \right)^N \left( \tanh \beta J \right)^N 2^N \]
\[ = \left( 2 \sinh \beta J \right)^N \]

Ben

Keep track of whether you kept 1 or \( \tanh \beta J \)
by shading in the bond <i,j> if you choose

\[ s_5 s_5 \tanh \beta J \]
In terms of this picture, what is condition for non-vanishing contribution? Well to save \( \sum S_5 = \phi \) needed to shade in 

\[ S_4 \ S_5 \]

safe.

Rule: cannot have free ends.

Only way to avoid free ends in \( d=1 \) is to form a closed loop by including all bonds.

High \( T \) expansion gives same answer as transfer matrix. Bonus is if gives \( Z \) for open bc

\( Z = (2 \cos \beta S)^N \)
can do transition matrix in $d=2$ but very hard (need eigenvalues of codim matrix!)

High $T$ in $d=2$ square

\[ Z = \sum \prod \cosh^T \left( \frac{1 + \sin \phi}{\sin \phi} \right) \]

Looks identical to $d=1$ except hidden in

is 2d square lattice topology

Same idea: largest contribution at high $T$

\[ Z = (\cosh^T)^{2N} 2^{2N} \quad \text{why} \quad 2^{2N}? \]

Pulling down just one bond gives $Z_2 = \text{as in d=1}$

\[ \text{free ends cause contribution to } Z \text{ to vanish} \]
How to avoid getting zero?

Need closed loop!

\[(\tanh \beta J)^4\]

There are \(N\) closed squares on lattice with \(N\) sites.

\[Z = (2 \cosh \beta J)^{2N} \left[ 1 + N (\tanh \beta J)^4 + \ldots \right] \]

Next:

\[+ 2N (\tanh \beta J)^6\]

Homework: \( (\tanh \beta J)^8 \) contribution.
\[(\tanh(\beta J))^8\]

\[
\begin{array}{c}
N \\
4N \\
2N
\end{array}
\]

"Disconnected diagrams will encounter in QFT"

\[
\frac{1}{2}N(N-5)
\]

\[
\frac{1}{2}N(N-5) + 7N
\]

\[\text{problem?}\]

\[
\frac{1}{2}N^2 + \frac{9}{2}N
\]

\[Z\text{ has } N^2\text{ term? } F = -kT \ln Z\]

\[\text{should be extensive } (\sim N^4)\]

Fortunately, cancels out \[\mp \tanh(\beta J)\]

\[
Z = (2\cosh(\beta J))^2N \left\{ 1 + N x^4 + 2N x^6 + \left( \frac{N^2}{2} + \frac{9N}{2} \right) x^8 + \ldots \right\}
\]

\[F = -kT \ln Z = -2NkT \ln(2\cosh(\beta J)) + \ln \left\{ \ldots \right\} \]

\[\ln \xi^2 = N x^4 + 2N x^6 + \left( \frac{N^2}{2} + \frac{9N}{2} \right) x^8 - \frac{1}{2} \left( N x^4 + \ldots \right)^2\]

\[\ln (1+x) = x - \frac{x^2}{2}\]

\[\frac{1}{2}N^2 + 8\]

cancels. This happens to all orders.
Spin-spin correlation functions in high T expansion:

\[ \langle A \rangle = z^{-1} \sum_{J} A e^{-\beta J} \leq \text{general} \]

\[ \langle s_{i}s_{j} \rangle = z^{-1} \sum_{1 \leq i < j \leq n} \frac{s_{i}s_{j} e^{\beta J}}{s_{1}s_{2},...s_{n}} \]

\[ = z^{-1} \sum_{s_{1}s_{2},...s_{n}} \frac{s_{i}(\cosh \beta J + s_{i}s_{j}\sinh \beta J)(\cosh \beta J + s_{i}s_{j}\sinh \beta J)}{s_{1}s_{2},...s_{n}} \]

Page: In computing \( z \), we took 1 from each factor to get biggest contribution at high \( T \) where \( \tanh \beta J \) is small.

\[ \langle s_{i}s_{i+1} \rangle = z^{-1} (\cosh \beta J)^{N} \sum_{s_{1},...,s_{N}} \frac{s_{i}(1 + s_{i}s_{j}t)(1 + s_{j}s_{i}t)}{s_{1}s_{2},...s_{n}} \]

What happens with \( s_{i} \) and \( s_{i+1} \)?

\[ \text{Answer: compare results} \]

Taking all 1's gives \( t \)!

get instead \( t \) as non-zero!

...
\[ <S, S_{\text{time}} > = (\tanh \beta J)^2 + (\tanh \beta J)^{N-2} \]

\[ <S, S_{\text{time}} > = e^{-\frac{1}{\xi}} \]

with \( \xi = \frac{1}{\ln(\tanh \beta J)} \) correlation length

- high \( T \), \( \beta \) small
- \( \tanh \beta J \) small
- \( \ln(\tanh \beta J) \) large and negative

\[ \xi = \frac{1}{\ln(\tanh \beta J)} \]

- low \( T \), \( \beta \) large, \( \tanh \beta J \approx 1 \)
- \( \ln(\tanh \beta J) \approx -\infty \)
- \( -\frac{1}{\ln(\tanh \beta J)} \approx 0 \)
$3$ diverges at phase transition \( s \sim \frac{1}{(T-T_c)^2} \)

Here in \( d=1 \) Ising it is \( I=0 \)

\[ \Rightarrow \text{No finite } T \text{ phase transition} \]

Physical significance of correlation function: