I 9

High temperature expansion method

$$e^{\beta S_i S_j} = \cosh\beta J + S_i S_j \sinh\beta J$$

for $S_i S_j = \pm 1$

$$= \cosh\beta J \left[ 1 + S_i S_j \tanh\beta J \right]$$

$$Z = \sum_{S_1 S_2 \ldots S_N} e^{\beta J \sum_{\langle i j \rangle} S_i S_j}$$

$$= \sum_{S_1 S_2 \ldots S_N} \prod_{\langle i j \rangle} e^{\beta J S_i S_j}$$

$$= \sum_{S_1 S_2 \ldots S_N} \prod_{\langle i j \rangle} \cosh\beta J \left( 1 + S_i S_j \tanh\beta J \right)$$

$$\doteq (\cosh\beta J)^{N} \sum_{S_1 S_2 \ldots S_N} \prod_{\langle i j \rangle} \left( 1 + S_i S_j \tanh\beta J \right)$$

N terms in $\prod_{\langle i j \rangle}$

Next geometrical construction:

On each bond you have your choice of

$1$ or $S_i S_j \tanh \beta J$ from $\prod_{\langle ij \rangle} (1 + S_i S_j \tanh \beta J)$

at high $T$ $\beta$ is small and so is $\tanh \beta J$ so

best to take $1$ all the time.

First term in high $T$ expansion

$$Z = (\cosh \beta J)^N \underbrace{\sum_{S_1 S_2 .. S_N} 1^N}_{2^N} \Rightarrow (2 \cosh \beta J)^N$$

Next biggest term: choose $1$ from all bonds

but one, suppose $S_5 S_6$

$$Z = (\cosh \beta J)^{N-1} \underbrace{\sum_{S_1 S_2 .. S_N} S_5 S_6 \tanh \beta J}_{\text{vanishes since } \sum_{S_5} S_5 = \phi}$$

only way to prevent $\sum_{S_5} S_5 = 0$

is to gather in another $S_5$ somehow $\sum_{S_5} S_5^2 = 2$

Eg

$$(\cosh\beta J)^{N-2} \sum_{S_1 S_2 \dots S_N} S_4 \underbrace{S_5 S_5}_{} S_6 (\tanh\beta J)^2$$

But $S_4$ $S_6$ continue to give $\phi$.

Must include $\underline{\underline{\text{all}}}$ the $S_i S_j \tanh\beta J$ and $\underline{\text{no}}$ $1$

to get non zero result. Only other contribution to $Z$

is $(\cosh\beta J)^N (\tanh\beta J)^N 2^N$

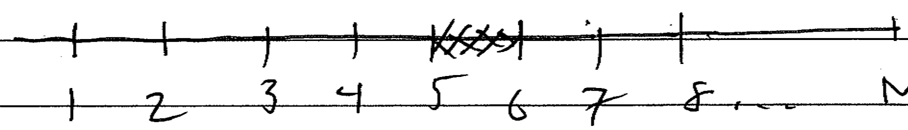
$= (2\sinh\beta J)^N$

Keep track of whether you'd kept $1$ or $\tanh\beta J$

by shading in the bond $\langle i\, j \rangle$ if you choose

$S_i S_j \tanh\beta J$

"diagram" for $S_5 S_6 \tanh\beta J$



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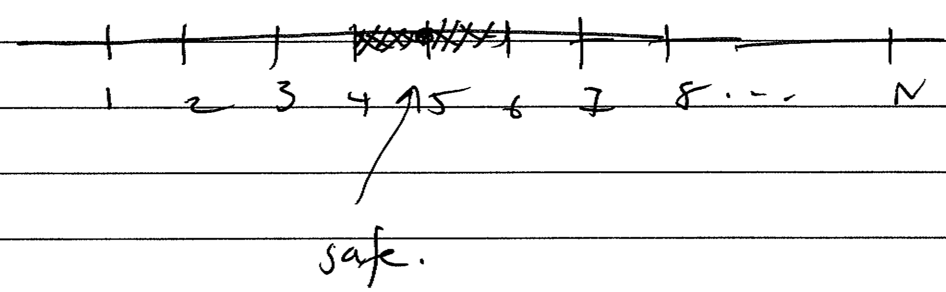
In terms of this picture, what is condition for non-vanishing contribution?

Well to save $\sum_{S_5} S_5 = \phi$ needed to shade in

$S_4 S_5$




Rule: cannot have free ends.

only way to avoid free ends in $d=1$ is to form a closed loop by including all bonds.

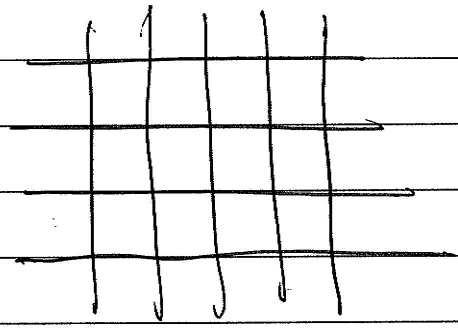
High T expansion gives same answer as transfer matrix. Bonus is it gives Z for open bc

$$Z = (2\cosh \beta J)^N$$

IB.

Can do transfer matrix in $d=2$ but very hard
(need eigenvalues of $\infty$ dim matrix!)

High $T$ in $d=2$ square



$N = \#$ sites

$$Z = \sum_{S_1 S_2 \ldots S_N} \prod_{\langle ij \rangle} \cosh\beta J \left(1 + S_i S_j \tanh\beta J\right)$$

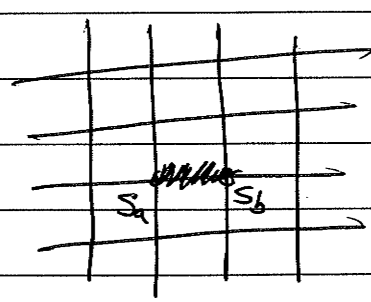
looks identical to $d=1$ except hidden in
is 2d square lattice topology

Same idea: largest contribution at high $T$

$$Z = (\cosh\beta J)^{2N} \, 2^{2N}$$

why $2N$?

Pulling down just one bond gives zero as in $d=1$

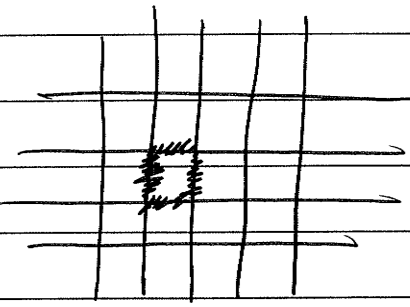



free ends cause
contribution to $Z$ to vanish

I/4

How to avoid getting zero?

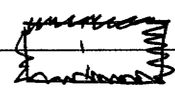
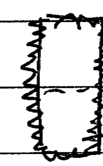
Need closed loop!



$(\tanh\beta J)^4$

There are $N$ closed squares on lattice with $N$ sites

$$Z = (2\cosh\beta J)^{2N}\left\{1 + N(\tanh\beta J)^4 + \dots\right\}$$

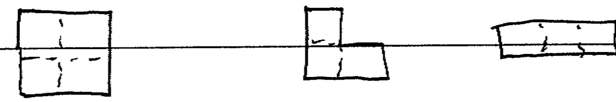
Next:  and 

$$+ 2N(\tanh\beta J)^6$$

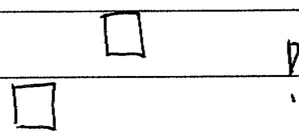
Homework: $(\tanh\beta J)^8$ contributions.

I14A

$(\tanh \beta J)^8$



$N \qquad 4N \qquad 2N$



"Disconnected diagrams" ← will encounter in QFT

$\frac{1}{2}N(N-5)$ problem?

$\frac{1}{2}N(N-5)+7N$

$\frac{1}{2}N^2+\frac{9}{2}N$

$Z$ has $N^2$ term!? $F=-kT\ln Z$ should be extensive $(\sim N^1)$

Fortunately, cancels out $\quad t \equiv \tanh\beta J$

$$Z=(2\cosh\beta J)^{2N}\left\{1+Nt^4+2Nt^6+\left(\frac{N^2}{2}+\frac{9N}{2}\right)t^8+\dots\right\}$$

$$F=-kT\ln Z=-2NkT\ln(2\cosh\beta J)+\ln\{\quad\}$$

$$\ln\{\}=Nt^4+2Nt^6+\left(\frac{N^2}{2}+\frac{9N}{2}\right)t^8-\frac{1}{2}\left(Nt^4+\dots\right)^2$$

$\ln(1+x)=x-\frac{x^2}{2}$

$\frac{1}{2}N^2t^8$

cancels. This happens to all orders!

I14B

Spin-Spin correlation functions in high T expansion; d=1

$$\langle A \rangle = Z^{-1} \sum A e^{-\beta E} \quad \leftarrow \text{general}$$

$$\langle S_1 S_{1+\ell} \rangle = Z^{-1} \sum_{S_1 S_2 \ldots S_N} S_1 S_{1+\ell} \, e^{\beta J \sum S_i S_j}$$

$$= Z^{-1} \sum_{S_1 S_2 \ldots S_N} S_1 \left(\cosh\beta J + S_1 S_2 \sinh\beta J\right)\left(\cosh\beta J + S_2 S_3 \sinh\beta J\right) \ldots$$

...

~~Pro~~ In computing $Z$ we took 1 from each ~~factor to get biggest contribution at high T where $\tanh\beta J$ is small~~

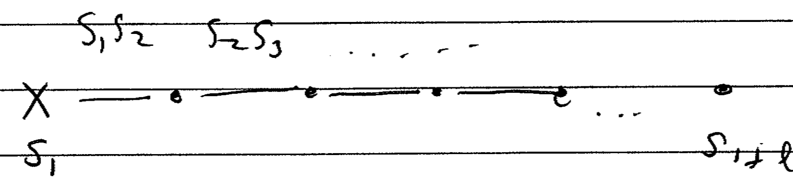
$$\langle S_1 S_{1+\ell} \rangle = Z^{-1} (\cosh\beta J)^N \sum_{S_1 S_2 \ldots} S_1 (1 + S_1 S_2 t)(1 + S_2 S_3 t) \ldots S_{1+\ell}$$

what happens with $S_1$ and $S_{1+\ell}$ there?

~~[illegible]~~

Taking all 1's gives $\phi$!

get instead $t^{\ell}$ as nonzero term!



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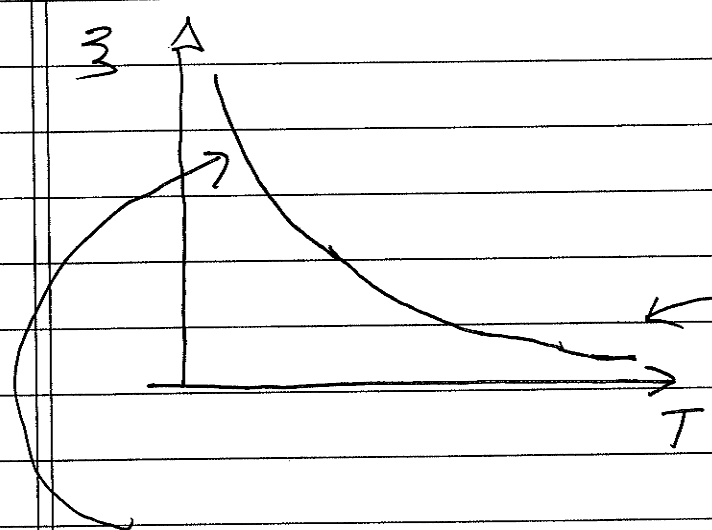
$$\langle S_1 S_{1+\ell} \rangle \simeq (\tanh\beta J)^{\ell} + (\tanh\beta J)^{N-\ell}$$

PBC term (drop, supposing $N-\ell \gg \ell$)

So

$$\langle S_1 S_{1+\ell} \rangle \simeq e^{-\ell/\xi}$$

with $\xi = -1/\ln(\tanh\beta J)$ correlation length




high T $\beta$ small

$\tanh\beta J$ small

$\ln(\tanh\beta J)$ large and negative

$\xi = -1/\ln(\tanh\beta J)$ small

low T $\beta$ large $\tanh\beta J \to 1$

$\ln(\tanh\beta J) \to 0^-$

$-1/\ln(\tanh\beta J) \to \infty$

14D

$\xi$ diverges at phase transition $\xi \sim \frac{1}{(T-T_c)^{\nu}}$

Here in d=1 Ising it is $T=0$

No finite T phase transition

Physical significance of correlation function: